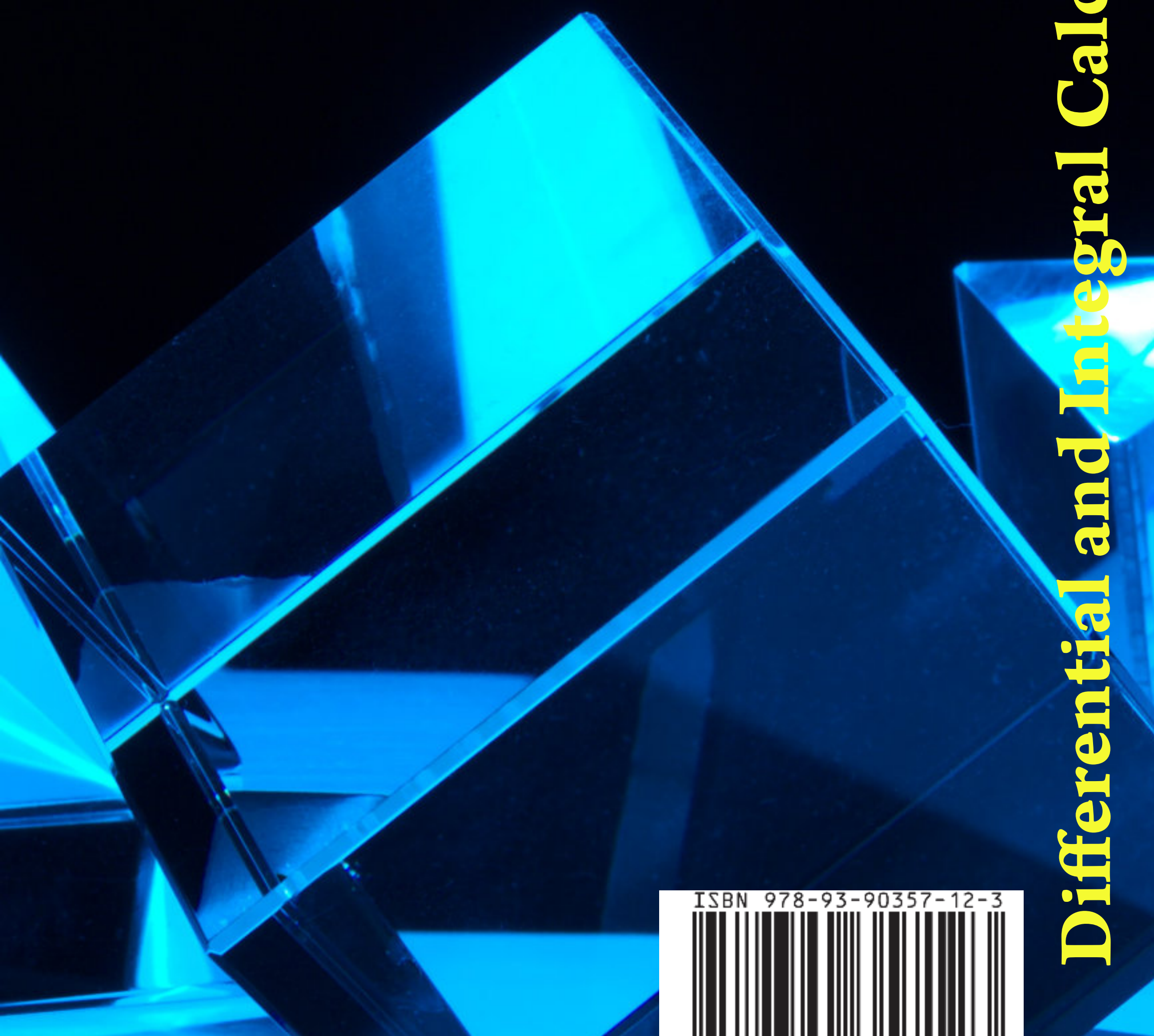
# Differential and Integral Calculus



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Differential and Integral Calculus

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# Differential and integral calculus

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad tg\alpha = \frac{\sin\alpha}{\cos\alpha}$$

$$ax^2 + bx + c = 0$$

$$a^2 - 2ab + b^2 \qquad (a-b)^2 = a$$

$$(a+b)(a-b) = a^2 - b^2$$

Ms. S. Asui Heris
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# Differential and integral calculus

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From the desk of

**Dr. T. X. A. ANANTH, BBA, MSW, MBA, MPhil, PhD,**

President – University Council

Dear Learner,

Welcome to DMI – St. Eugene University!

I am sure you are expert in using the PC Tablets distributed by us. Now your world is open to Internet and using the tablet for your educational learning purposes. The very same book you are holding in your hand now is available in your V-Campus portal. All the teaching and learning materials are available in your portal.

As our Chancellor, Rev. Fr. Dr. J. E. Arulraj , mentioned, it is not just the success for DMI-St. Eugene University alone, it is success for the technology, it is success for the great nation of Zambia and it is success for the continent of Africa.

You can feel the improvement in the quality of the learning materials in the printed format. Improvement is done in quality of the content and the printing. Use it extensively and preserve it for your future references. This will help you to understand the subjects in a better way. The theories of the subjects have been explained thoroughly and the problems have been identified at learners' level. This book is made up of five units and every unit is mapped to the syllabus and discussed in detail.

I am happy at the efforts taken by the University in publishing this book not only in printed format, but also in PDF format in the Internet.

With warm regards

Dr. T. X. A. ANANTH

President – University Council

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# UNIT- I

## LIMITS AND CONTINUITY, DIFFERENTIATION

## LIMITS AND CONTINUTIY

### 1.0 Introduction

In this chapter we consider functions whose domain and range are subsets of **R** and we consider the limit of such functions. This concept of limit is basic to the study of continuous functions and differential calculus.

The concept of differential coefficient and the operation called differentiation are basic to the theory of differential calculus this chapter is devoted to the study of differentiation and the algebra of derivatives.

### 1.1 Limits of a function

**Definition**

Let$A \subseteq \boldsymbol{R}$**.** A function $\boldsymbol{f}: \boldsymbol{A} \rightarrow \boldsymbol{R}$ is called a real valued function of real variable.

Throughout this chapter we shall be concerned with such functions only and in most cases the domain of the function is restricted to an interval in **R**. It may happen that for a function $f$ as $x$ approaches closer and closer to $a$ the value $f(x)$ approaches closer and closer to a definite real number$l$. For example if $f(x) = x^2 + 1$ then as $x$ approaches closer to closer to 2, $f(x)$ approaches closer and closer to 5. We say that the limit of $f(x) = x^2 + 1$ as tends to 2 is 5, and we write $\lim_{x \to 2}(x^2 + 1) = 5$.

Now, consider the function $f(x) = \frac{x^2-1}{x-1}$ . We proceed to investigate what happens when $x$ approaches 1. In this case both numerator and denominator approach 0. It would be meaningless to say that the function approaches $\frac{0}{0}$ since $\frac{0}{0}$ is not a symbol for any number.

However $\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$.

Also we are concerned with what is happening to the function as $x$ approaches 1 and not with what happens when $x = 1$. Moreover while $x$ approaches 1, $\frac{x^2-1}{x-1}$ and $x + 1$ have exactly the same values. Further $x + 1$ approaches 2 as $x$ approaches 1.

Hence $\frac{x^2-1}{x-1}$ approaches 2 as $x$ approaches 1.

Thus $\lim_{x\to 1}\frac{x^2-1}{x-1} = 2$.

The precise meaning of this concept of limit is given in the following definition.

**Definition**

A function $f$ is said to approach to a limit $l$ as $x$ tends to $a$ if given $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x-a| < \delta \Rightarrow |f(x) - l| < \varepsilon$ and we write $\lim_{x\to a} f(x) = l$.

**Note**

1. It should be carefully noted that the condition $0 < |x-a| < \delta$ excludes the point $x = a$ from consideration. Hence the definition of limit employs only values of $x$ in some interval $(a-\delta, a+\delta)$ other than $a$. Hence the value of $f(x)$ at $x = a$ is immaterial and in fact to consider $\lim_{x\to a} f(x)$ , $f(x)$ need not be even defined at $x = a$. Even if $f(a)$ is defined it is not necessary that $\lim_{x\to a} f(x) = f(a)$. (refer example 3 below).

2. To talk about $\lim_{x\to a} f(x)$, it is necessary that the domain of definition of the function f must contain the set $(a-\delta, a+\delta) - \{a\}$ for some $\delta > 0$. A subset **A** of **R** containing an interval of the form $(a-\delta, a+\delta)$ for some $\delta > 0$ is called a neighborhood of $a$. Thus to talk about $\lim_{x\to a} f(x)$ it is necessary that $f(x)$ must be defined in some neighbourhood of $a$ except perhaps at $a$.

**Example 1**

$\lim_{x\to a} kx = ka$ where $k$ is any non-zero real number.

**Solution**

Let $\varepsilon > 0$ be given. Then $|kx - ka| = |k||x - a|$. Now choose $\delta = \frac{\varepsilon}{|k|}$.

$\therefore\ 0 < |x-a| < \frac{\varepsilon}{|k|} \Rightarrow |kx - ka| < |k|\frac{\varepsilon}{|k|} = \varepsilon$

$\therefore\ \lim_{x\to a} kx = ka$.

**Example 2**

$\lim_{x\to 0} x^2 = 0$.

**Solution**

Let $\varepsilon > 0$ be given.

Then $0 < |x| < \sqrt{\varepsilon} \Rightarrow |x^2 - 0| = x^2 < \varepsilon.$

$\therefore \lim\limits_{x\to 0} x^2 = 0$

**Example 3**

Let $f(x) = \begin{cases} 2x+3 & if\ x \neq 1 \\ 10 & if\ x = 1 \end{cases}$. Then $\lim\limits_{x\to 1} f(x) = 5.$

**Solution**

Let $\varepsilon > 0$ be given.

Now $|2x + 3 - 5| = |2x - 2| = 2|x - 1|$.

Choose $\delta = \frac{1}{2}\varepsilon$.

Then $0 < |x - 1| < \frac{1}{2}\varepsilon \Rightarrow |(2x + 3) - 5| < 2\left(\frac{1}{2}\varepsilon\right) = \varepsilon$

$\therefore \lim\limits_{x\to 1} f(x) = 5$

Note that here $f(1) = 10$ so that $\lim\limits_{x\to 1} f(x) \neq f(1)$.

**Example 4**

$\lim\limits_{x\to 2} \frac{x^2-4}{x-2} = 4.$

**Solution**

Let $\varepsilon > 0$ be given.

Now, when $x \neq 2, \frac{x^2-4}{x-2} = x + 2.$

$\therefore \left|\frac{x^2-4}{x-2} - 4\right| = |x + 2 - 4| = |x - 2|.$

$\therefore$ If we choose $\delta = \varepsilon$, then $0 < |x - 2| < \delta \Rightarrow \left|\frac{x^2-4}{x-2} - 4\right| < \varepsilon$

$\therefore \lim\limits_{x\to 2} \left(\frac{x^2-4}{x-2}\right) = 4.$

**Exercise 1**

Prove the following:

1. $\lim\limits_{x\to 2}(3x - 4) = 2$
2. $\lim\limits_{x\to \frac{1}{2}}(1 - x) = \frac{1}{2}$
3. $\lim\limits_{x\to 3} \frac{x^2-9}{x-3} = 6$
4. $\lim\limits_{x\to -2} \frac{x^2+x-2}{x+2} = -3$
5. $\lim\limits_{x\to a}(\alpha x + \beta) = \alpha\, a + \beta$

6. Let $\boldsymbol{f}: \boldsymbol{R} \to \boldsymbol{R}$ be defined by $f(x) = \begin{cases} x & if\ x \neq 0 \\ 2 & if\ x = 0 \end{cases}$. Prove that $\lim_{x \to 0} f(x) = 0$.

We state without proof the following theorems on limits.

**Theorem 1.1**

If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} f(x) = m$, then $l = m$, (i.e.) the limit of $f(x)$ as $x \to a$, if it exists, is unique.

**Theorem 1.2**

If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then $\lim_{x \to a}[f(x) + g(x)] = l + m$.

**Theorem 1.3**

If $\lim_{x \to a} f(x) = l$, then $\lim_{x \to a} k\, f(x) = kl$ where $k$ is any real number.

**Theorem 1.4**

If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then $\lim_{x \to a}[f(x) - g(x)] = l - m$.

**Theorem 1.5**

If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then $\lim_{x \to a} f(x)g(x) = lm$.

**Theorem 1.6**

If $\lim_{x \to a} f(x) = l$ and $f(x) \neq 0$ and $l \neq 0$, then $\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{l}$.

**Theorem 1.7**

If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$ where $m \neq 0$ and $g(x) \neq 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m}$.

**Theorem 1.8**

If $\lim_{x \to a} f(x) = l$, then $\lim_{x \to a} |f(x)| = |l|$.

**Theorem 1.9**

Let $f$ be a bounded function. Let $\lim_{x\to a} g(x) = 0$ . Then $\lim_{x\to a} f(x)g(x) = 0$.

**Theorem 1.10**

If $f(x) \geq 0$ and $\lim_{x\to a} f(x) = l$, then $l \geq 0$.

**Theorem 1.11**

If $f(x) \leq g(x)$ and $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, then $l \leq m$.

**Theorem 1.12**

If $g(x) \leq f(x) \leq h(x)$ and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = l$ , then $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) = l$.

**Theorem 1.13**

$\lim_{\theta\to 0} \sin\theta = 0.$

**Theorem 1.14**

$\lim_{x\to 0} \cos x = 1.$

**Theorem 1.15**

$\lim_{x\to a} \sin x = \sin a.$

**Theorem 1.16**

$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

**Theorem 1.17**

$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e.$

**Theorem 1.18**

$\lim_{x\to 0} \frac{\tan x}{x} = 1.$

**Example 5**

Evaluate $\lim_{x\to 2}(3+2x+5x^2+6x^3)$.

**Solution**

$$\begin{aligned}\lim_{x\to 2}(3+2x+5x^2+6x^3) &= \lim_{x\to 2} 3 + \lim_{x\to 2} 2x + \lim_{x\to 2} 5x^2 + \lim_{x\to 2} 6x^3 \\ &= 3 + 2\left(\lim_{x\to 2} x\right) + 5\left(\lim_{x\to 2} x^2\right) + 6\left(\lim_{x\to 2} x^3\right) \\ &= 3+4+20+48 = 75.\end{aligned}$$

**Example 6**

If $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, prove that $\lim_{x\to a} f(x) = f(a)$.

**Solution**

$$\begin{aligned}\lim_{x\to a} f(x) = \lim_{x\to a}(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) &= \lim_{x\to a} a_0 + \lim_{x\to a} a_1 x + \cdots + \lim_{x\to a} a_n x^n \\ &= a_0 + a_1\left(\lim_{x\to a} x\right) + \cdots + a_n\left(\lim_{x\to a} x^n\right) \\ &= a_0 + a_1 a + \cdots + a_n a^n = f(a).\end{aligned}$$

**Example 7**

Evaluate $\lim_{x\to 2}\left(\frac{3x+4}{x-3}\right)$

**Solution**

$\lim_{x\to 2}(x-3) = -1 \neq 0.$

$$\therefore \lim_{x\to 2}\frac{3x+4}{x-3} = \frac{\lim_{x\to 2}(3x+4)}{\lim_{x\to 2}(x-3)} = \frac{10}{-1} = -10.$$

**Example 8**

Evaluate $\lim_{x\to 1}\frac{x^3+x^2-x-1}{x-1}$.

**Solution**

When $x \neq 1$, $\frac{x^3+x^2-x-1}{x-1} = \frac{(x-1)(x^2+2x+1)}{x-1} = x^2+2x+1.$

$$\therefore \lim_{x\to 1}\frac{x^3+x^2-x-1}{x-1} = \lim_{x\to 1}(x^2+2x+1) = 1+2+1 = 4.$$

**Example 9**

Show that $\lim_{x \to 0} \left[ x \sin\left(\frac{1}{x}\right) \right] = 0$.

**Solution**

Let $f(x) = \sin\left(\frac{1}{x}\right)$ and $g(x) = x$.

Clearly $|f(x)| \leq 1$

$\therefore$ $f$ is a bounded function.

Also $\lim_{x \to 0} g(x) = \lim_{x \to 0} x = 0$.

$\therefore$ $\lim_{x \to 0} \left[ x \sin\left(\frac{1}{x}\right) \right] = 0$ (by theorem 1.9)

**Exercise 2**

1. Evaluate the following limits:

(i) $\lim_{x \to 1} (5x^2 + 3x - 2)$ (ii) $\lim_{x \to 0} (2x^3 - 5x)$

(iii) $\lim_{x \to 0} \frac{x^2+2x+5}{x^2+1}$ (iv) $\lim_{x \to 2} \frac{x-2}{\sqrt{2+x}}$

(v) $\lim_{x \to 0} \frac{4x^3-2x^2+x}{3x^2+2x}$ (vi) $\lim_{x \to 2} \frac{x^2+3x-10}{3x^2-5x-2}$

(vii) $\lim_{x \to 1} \frac{x^3-1}{x-1}$ (viii) $\lim_{x \to 2} \frac{x^3-5x+6}{x^2-12x+20}$

(ix) $\lim_{y \to -2} \frac{y^3-3y^2+2y}{y^2-y-6}$ (x) $\lim_{u \to -2} \frac{u^3+4u^2+4u}{(u+2)(u-3)}$

(xi) $\lim_{x \to 0} \left[ x \cos x \left(\frac{1}{x}\right) \right]$

2. Let $f(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m$ and $g(x) = b_0 x^n + b_1 x^{n-1} + \cdots + b_n$. Prove that $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$ if $g(a) \neq 0$.

## 1.2 Left and Right limits

While defining the limit of $f(x)$ as $x \to a$, we consider the behavior of $f(x)$ at points which are near to $a$ and these points can be either to the left of $a$ or to the right of $a$. However it is often necessary to know the behavior of $f(x)$ as $x$ tends to $a$ in such a way that $x$ always remains greater than (or less than) $a$. This leads us to the concept of right and left limits of $f(x)$ at $x = a$.

**Definition**

A function $f$ is said to have $l$ as the right limit at $x = a$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < x - a < \delta \Rightarrow |f(x) - l| < \varepsilon$ and we write $\lim_{x \to a+} f(x) = l$. Also we denote the right limit $l$ at $a$ by $f(a+)$

A function $f$ is said to have $l$ as the left limit at $x = a$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < a - x < \delta \Rightarrow |f(x) - l| < \varepsilon$ and we write $\lim_{x \to a-} f(x) = l$. Also we denote the left limit $l$ at $a$ by $f(a-)$

**Theorem 1.19**

$\lim_{x \to a} f(x) = l$ iff $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = l$.

(i.e.) $\lim_{x \to a} f(x)$ exists iff the left limit and the right limit of $f(x)$ at $x = a$ exist and are equal.

**Note**

If $\lim_{x \to a} f(x)$ does not exist, then one of the following happens.

(i) $\lim_{x \to a+} f(x)$ does not exist.

(ii) $\lim_{x \to a-} f(x)$ does not exist.

(iii) $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$ exist and are not equal.

Thus the concepts of left limit and right limit can be used in many cases to prove the nonexistence of limit.

**Example 10**

Let $\boldsymbol{f}: \boldsymbol{R} \to \boldsymbol{R}$ be defined by $f(x) = \begin{cases} 0 \ if \ x \leq 0 \\ 1 \ if \ x > 0 \end{cases}$

Then $\lim_{x \to 0+} f(x) = 1$ and $\lim_{x \to 0-} f(x) = 0$.

**Example 11**

Let $\boldsymbol{f}: \boldsymbol{R} \to \boldsymbol{R}$ be defined by $f(x) = [x]$ where $[x]$ is the integral part of $x$. Then for any integer n, $\lim_{x \to n} f(x)$ does not exist, since $\lim_{x \to n+} f(x) = n$ and $\lim_{x \to n-} f(x) = n - 1$.

**Exercise 3**

1. Let $f: [1, 3] \to \boldsymbol{R}$ be defined by $f(x) = \begin{cases} x^2 & if\ 1 \le x \le 2 \\ 3x - 4 & if\ 2 \le x \le 3 \end{cases}$. Show that $f(2+) = 2$ and $f(2-) = 4$.

2. Let $f: [0, 1] \to \boldsymbol{R}$ be defined by $f(x) = \begin{cases} x & if\ 0 \le x \le \frac{1}{2} \\ 1 - x & if\ \frac{1}{2} \le x \le 1 \end{cases}$. Show that $f(0+) = 0$ and $f(1-) = 0$.

## 1.3 Continuous Functions

**Definition**

A function $f$ is said to be continuous at $x = a$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

If a function $f$ is not continuous at $a$ then $f$ is said to be discontinuous at $a$.

A function $f$ is said to be continuous if it is continuous at every point of its domain.

**Note**

1. $f$ is continuous at $x = a$ iff $\lim_{x \to a} f(x)$ exists and is equal to $f(a)$.

2. If a function $f$ is defined on a closed interval $[a, b]$, then at the end point $a$ we can only talk about the right limit of $f(x)$ and similarly at the end point $b$ we can only talk about the left limit of $f(x)$. Hence the continuity of $f$ at the end points $a$ and $b$ are defined by the conditions $f(a) = \lim_{x \to a+} f(x)$ and $f(b) = \lim_{x \to b-} f(x)$.

**Example 12**

Let $\boldsymbol{f}: \boldsymbol{R} \to \boldsymbol{R}$ be defined by $f(x) = x + 3$. Then $f$ is continuous at every point $a \in \boldsymbol{R}$.

**Solution**

For, $\lim_{x \to a} f(x) = \lim_{x \to a}(x + 3) = a + 3 = f(a)$.

**Example 13**

Let $\boldsymbol{f}: \boldsymbol{R} \to \boldsymbol{R}$ be defined by $f(x) = kx$. Then $f$ is continuous at every point $a \in \boldsymbol{R}$.

**Solution**

For, $\lim_{x \to a} f(x) = \lim_{x \to a} kx = ka = f(a)$.

**Example 14**

Any polynomial function given by $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is continuous at every point $a \in \boldsymbol{R}$. (refer example 6)

**Example 15**

Let $\boldsymbol{f}: \boldsymbol{R} \to \boldsymbol{R}$ be defined by $f(x) = \sin x$. Then by theorem 1.15, $f$ is continuous at every point $a \in \boldsymbol{R}$.

**Example 16**

Let $\boldsymbol{f}: \boldsymbol{R} \to \boldsymbol{R}$ be defined by $f(x) = |x|$. Then $f$ is continuous at every point $a \in \boldsymbol{R}$.

**Solution**

For, $\lim_{x \to a} f(x) = \lim_{x \to a} |x| = |a| = f(a)$.

**Example 17**

Let $\boldsymbol{f}: \boldsymbol{R} \to \boldsymbol{R}$ be defined by $f(x) = [x]$. Then $f$ is not continuous at each integer n. For $\lim_{x \to n} f(x)$ does not exist. (refer example 11).

**Exercise 4**

1. Show that any constant function is continuous at every point.
2. Show that the identity function $\boldsymbol{f}: \boldsymbol{R} \to \boldsymbol{R}$ defined by $f(x) = x$ is continuous at every point.

**Remark**

(i) If $f$ and $g$ are continuous at $a$ then $f + g$ is continuous at $a$.

(ii) If $f$ and $g$ are continuous at $a$ then $fg$ is continuous at $a$.

(iii) If $f$ and $g$ are continuous at $a$ and $g(a) \neq 0$ then $(f/g)$ is continuous at $a$.

(iv) If $f$ is continuous at $a$ then $|f|$ is continuous at $a$.

(v) If $f$ is continuous at $a$ and $g$ is continuous at $f(a)$, then $g \circ f$ is continuous at $a$. (i.e.) Continuous function of a continuous function is continuous.

**Example 18**

$f(x) = x$ is continuous at every point. Hence $\frac{1}{x}$ is continuous at every point $x \neq 0$.

Also $\frac{1}{x}$ is not defined at $x = 0$.

$\therefore \frac{1}{x}$ is not continuous at $x = 0$.

**Example 19**

$f(x) = x^2 + 1$ is continuous at every point.

Also $x^2 + 1 \neq 0$ for all $a \in \boldsymbol{R}$.

$\therefore \frac{1}{(x^2+1)}$ is continuous at every point.

**Example 20**

$f(x) = \tan x$ is continuous at all point except at $x = (2n+1)\frac{\pi}{2}, n \in \boldsymbol{Z}$.

For, $f(x) = \frac{\sin x}{\cos x}$

Now, $\sin x$ and $\cos x$ are continuous at all points. Also $\cos x = 0$ where $x = (2n+1)\frac{\pi}{2}, n \in \boldsymbol{Z}$.

$\therefore \tan x$ is not continuous at these points.

At all other points $\tan x$ is continuous.

**Example 21**

$f(x) = \sin 2x$ is continuous at all points since, $\sin 2x = 2\sin x \cos x$ which is a product of continuous functions.

**Example 22**

Let $f(x) = \sin x$ and $g(x) = \frac{1}{x}$. Then $f$ is continuous at all points and $g$ is continuous at all points $x \neq 0$.

$\therefore \left(f \circ g(x)\right) = f\left(g(x)\right) = f\left(\frac{1}{x}\right) = \sin\left(\frac{1}{x}\right)$ is continuous at all points $x \neq 0$.

**Example 23**

Let $f(x) = \sin x$ and $g(x) = x^2$.

$\therefore f$ and $g$ are continuous at all points.

$\therefore \left(g \circ f(x)\right) = g\left(f(x)\right) = g(\sin x) = \sin^2 x.$

**Definition**

If a function $f$ is discontinuous at $a$, then $a$ is called a point of discontinuity for the function.

**Note**

If $a$ is a point of discontinuity for a function then any one of the following cases arise.

(i) $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$.

(ii) $\lim_{x \to a} f(x)$ does not exist.

(i.e.) (a) Either $\lim_{x \to a+} f(x)$ or $\lim_{x \to a-} f(x)$ does not exist.

or (b) $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$ exist but are not equal.

**DIFFERENTIATION**

## 1.4 Differentiability

**Definition**

Let $f$ be a function defined on an open interval $I$ in **R**. Let $x \in I$. We say that $f$ is differentiable at x if $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ exists and is finite.

The value of the above limit is called the differential coefficient or derivative of $f$ with respect to $x$ and it is denoted by $f'(x)$ or $\frac{df}{dx}$ or $\frac{dy}{dx}$ or $y'$ where $y = f(x)$.

If $\lim_{h \to 0+} \frac{f(x+h)-f(x)}{h}$ exists and is finite, we say that $f$ is differentiable from the right at $x$. The value of the above right limit is denoted by $R\ f'(x)$ and is called the right derivative of $f$ at $x$.

Similarly the left derivative $L\ f'(x)$ can be defined.

**Note**

1. f is differentiable at x iff f is left differentiable and right differentiable at x and $L\ f'(x) = R\ f'(x)$

2. If $f$ is differentiable at every point of an open interval $I$ then we say that $f$ is differentiable on $I$.

3. If f is defined on $[a, b]$, then $f$ is said to be differentiable on $[a, b]$, if it is differentiable on $(a, b)$, right differentiable at $a$ and left differentiable at $b$.

**Theorem 1.20**

If $f$ is differentiable at $x$ then $f$ is continuous at $x$.

**Proof**

Since $f$ is differentiable at $x$, we have $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists and is finite.

Now, $f(x+h) - f(x) = \left[\frac{f(x+h)-f(x)}{h}\right] h.$

$$\therefore \lim_{h\to 0}[f(x+h) - f(x)] = \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h}\right] h$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \lim_{h\to 0} h$$

$$= f'^{(x)} \times 0 = 0.$$

$\therefore \lim_{h\to 0} f(x+h) = f(x)$

$\therefore f$ is continuous at $x$.

**Note:** The converse of the above theorem is not true. (i.e.) A function $f$ which is continuous at $x$ need not be differentiable at $x$.

**Example 24**

Consider $f(x) = |x|$.

**Solution**

$\lim_{x\to 0} f(x) = \lim_{x\to 0}|x| = 0 = f(0)$

$\therefore f$ is continuous at $x = 0$.

We note that $f(x) = \begin{cases} x & if\ x > 0 \\ -x & if\ x < 0 \end{cases}$

$$\therefore R\, f'(0) = \lim_{h\to 0+} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0+} \frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0} \left(\frac{h-0}{h}\right) \qquad (\because h > 0)$$

$$= 1$$

Also $L\, f'(0) = \lim_{h\to 0-} \frac{f(h)-f(0)}{h}$

$$= \lim_{h\to 0} \left(\frac{-h-0}{h}\right) \qquad (\because h < 0)$$

$$= -1$$

$\therefore R\, f'(0) \neq L\, f'(0)$

$\therefore f$ is not differentiable at $x = 0$.

**Example 25**

Consider $f(x) = \begin{cases} 2+x & if\ x \geq 0 \\ 2-x & if\ x < 0 \end{cases}$

**Solution**

$\lim_{x\to 0+} f(x) = \lim_{x\to 0}(2+x) = 2.$

$\lim_{x\to 0-} f(x) = \lim_{x\to 0}(2-x) = 2.$

$\therefore \lim_{x\to 0} f(x) = 2 = f(0).$

$\therefore$ $f$ is continuous at $x = 0$.

Now, $R\ f'(0) = \lim_{h\to 0+} \frac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0+} \frac{f(h)-f(0)}{h}$

$= \lim_{h\to 0} \frac{2+h-2}{h} = 1$

Also, $L\ f'(0) = \lim_{h\to 0-} \frac{f(h)-f(0)}{h}$

$= \lim_{h\to 0} \frac{(2-h)-2}{h} = -1$

$\therefore R\ f'(0) \neq L\ f'(0)$

$\therefore$ $f$ is not differentiable at $x = 0$.

## 1.5 Algebra of derivatives

**Theorem 1.21**

Let $f(x) = u(x) + v(x)$. Let $u(x)$ and $v(x)$ be differentiable at $x$. Then $f(x)$ is also differentiable at $x$ and $(u+v)'(x) = u'(x) + v'(x)$.

**Proof**

$f(x) = u(x) + v(x)$

$\therefore f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \left[\frac{u(x+h)+v(x+h)-[u(x)+v(x)]}{h}\right]$

$= \lim_{h\to 0} \left[\frac{u(x+h)-u(x)}{h} + \frac{v(x+h)-v(x)}{h}\right]$

$= \lim_{h\to 0} \left[\frac{u(x+h)-u(x)}{h}\right] + \lim_{h\to 0} \left[\frac{v(x+h)-v(x)}{h}\right]$

$= u'(x) + v'(x)$

**Theorem 1.22**

Let $u(x)$ be differentiable at $x$ and $c \in \boldsymbol{R}$. Then $c\ u(x)$ differentiable at $x$ and $(cu)'(x) = c\ u'(x)$

**Proof**

Let $f(x) = c\,u(x)$.

$$\therefore f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{c\,u(x+h)-c\,u(x)}{h}$$
$$= c\,u'(x)$$

**Theorem 1.23**

Let $f(x) = u(x)v(x)$. Let $u(x)$ and $v(x)$ are differentiable, then $f(x)$ is also differentiable and $(uv)'(x) = u(x)v'(x) + v(x)u'(x)$.

**Proof**

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{u(x+h)v(x+h)-u(x)v(x)}{h}$$
$$= \lim_{h\to 0}\frac{u(x+h)v(x+h)-u(x+h)v(x)+u(x+h)v(x)-u(x)v(x)}{h}$$
$$= \lim_{h\to 0}u(x+h)\left[\frac{v(x+h)-v(x)}{h}\right] + \lim_{h\to 0}v(x)\left[\frac{u(x+h)-u(x)}{h}\right] =$$
$$u(x)v'(x) + v(x)u'(x).$$

**Note**

The above result can be extended to a product of n functions which are differentiable, as follows.

$(u_1, u_2, \dots, u_n)' = u_1'u_2 \dots u_n + u_1u_2'u_3 \dots u_n + \cdots + u_1u_2 \dots u_n{}'$.

This result can be proved by induction.

**Theorem 1.24**

Let $f(x) = \frac{u(x)}{v(x)}$. Let $u(x)$ and $v(x)$ are differentiable and $v(x) \neq 0$, then $f$ is differentiable and $\left(\frac{u}{v}\right)'(x) = \frac{v(x)u'(x)-u(x)v'(x)}{[v(x)]^2}$.

**Proof**

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}\right]$$
$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{v(x)u(x+h)-v(x+h)u(x)}{v(x+h)v(x)}\right]$$
$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{v(x)u(x+h)-v(x)u(x)+u(x)v(x)-v(x+h)u(x)}{v(x+h)v(x)}\right]$$

$$= \lim_{h\to 0} \frac{1}{v(x+h)v(x)} \left[ v(x) \left( \frac{u(x+h)-u(x)}{h} \right) - u(x) \left( \frac{v(x+h)-v(x)}{h} \right) \right]$$
$$= \frac{v(x)u'(x)-u(x)v'(x)}{[v(x)]^2}$$

**Using the algebra of derivatives we can derive the derivative of any differentiable function.**

**1.6 Derivative of Standard functions**

**Derivatives of some standard functions without proof**

**Result 1:** Let $f(x) = c$ be a constant function. Then $\frac{dc}{dx} = 0$.

**Result 2:** $\frac{d}{dx}(\sin x) = \cos x$.

**Result 3:** $\frac{d}{dx}(\text{ocs}\, x) = -\sin x$.

**Result 4:** $\frac{d}{dx}(x^n) = nx^{n-1}$.

**Result 5:** $\frac{d}{dx}(e^x) = e^x$.

**Result 6:** $\frac{d}{dx}(\log x) = \frac{1}{x}$.

**Result 7:** $\frac{d}{dx}(\tan x) = \sec^2 x$ if $x \neq (2n+1)\frac{\pi}{2}$.

**Result 8:** $\frac{d}{dx}(\sec x) = \sec x \tan x$ if $x \neq (2n+1)\frac{\pi}{2}$.

**Result 9:** $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$ if $x \neq n\pi$.

**Result 10:** $\frac{d}{dx}(\cot x) = -\text{cosec}^2\, x$ if $x \neq n\pi$.

**Hyperbolic functions**

The hyperbolic functions are defined by

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

The other hyperbolic functions $\tanh x$, $\text{cosech}\, x$, $\text{sech}\, x$ and $\coth x$ are defined in terms of $\sinh x$ and $\cosh x$ as in the case of circular trigonometric functions.

**Results**

1. $\cosh^2 x - \sinh^2 x = 1$.
2. $\sinh 2x = 2 \sinh x \cosh x$.
3. $\cosh^2 x + \sinh^2 x = \cosh 2x$.
4. $1 - \tanh^2 x = \text{sech}^2\, x$.

5. $1-\coth^2 x = -\operatorname{cosech}^2 x$.

**Result 11:** $\frac{d}{dx}(\sinh x) = \cosh x$.

**Result 12:** $\frac{d}{dx}(\cosh x) = \sinh x$.

**Result 13:** $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

**Result 14:** $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

**Result 15:** $\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$

**Result 16:** $\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$

## 1.7 The chain rule for differentiation

**Theorem 1.25**

Let $f:[a,b] \to [c,d]$ and $g:[c,d] \to [e,f]$ be two continuous functions. Suppose $f$ is differentiable at $x \in (a,b)$ and $g$ is differentiable at $y = f(x) \in (c,d)$, then $g \circ f$ is differentiable at x and $(g \circ f)'(x) = g'\big(f(x)\big)f'(x)$.

**Proof**

We have to prove that $\lim_{h\to 0} \frac{(g\circ f)(x+h)-(g\circ f)(x)}{h}$ exists and is equal to $g'\big(f(x)\big)f'(x)$.

Now, $\frac{(g\circ f)(x+h)-(g\circ f)(x)}{h} = \left[\frac{(g\circ f)(x+h)-(g\circ f)(x)}{f(x+h)-f(x)}\right]\left[\frac{f(x+h)-f(x)}{h}\right]$

Let $f(x+h) = y+k$ and $f(x) = y$.

Since $f$ is continuous, $k \to 0$ as $h \to 0$.

Now, $\frac{(g\circ f)(x+h)-(g\circ f)(x)}{h} = \left[\frac{g(y+k)-g(y)}{k}\right]\left[\frac{f(x+h)-f(x)}{h}\right]$

Taking limit as $h \to 0 (k \to 0)$ and using the fact that $f$ is differentiable at $x$ and $g$ is differentiable at $f(x)$ we get $(g \circ f)'(x) = g'\big(f(x)\big)f'(x)$.

**Example 26**

Find the derivatives of the following functions w. r. t. $x$.

1. $\sin 2x$ 2. $\sin^2 x$ 3. $\sin x^2$

4. $\sin\sqrt{x}$ 5. $\sqrt{\sin x}$ 6. $\sqrt{\left(\sin\sqrt{x}\right)}$

7. $\sin(\sin x)$ 8. $\sin\left(\sin\sqrt{x}\right)$ 9. $\sin(\log x)$

10. $\log(\sin x)$ 11. $e^{\sin x}$ 12. $\sin e^x$ 13. $\sin x^\circ$

**Solution**

**1.** Let $y = \sin 2x$

Let $f(x) = 2x$ and $g(x) = \sin x$.

$\therefore \frac{dy}{dx} = g'\big(f(x)\big)f'(x) = (\cos 2x)2 = 2\cos 2x.$

**2.** Let $y = \sin^2 x$.

$\therefore \frac{dy}{dx} = 2\sin x \cos x = \sin 2x.$

**3.** Let $y = \sin(x^2)$

$\therefore \frac{dy}{dx} = \cos(x^2)\ 2x = 2x\cos(x^2)$

**4.** Let $y = \sin\sqrt{x}$

$\therefore \frac{dy}{dx} = \cos\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{x}}\cos\sqrt{x}$

**5.** Let $y = \sqrt{(\sin x)}$

$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{(\sin x)}}\cos x = \frac{\cos x}{2\sqrt{(\sin x)}}.$

**6.** Let $y = \sqrt{\left(\sin\sqrt{x}\right)}$

$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{\left(\sin\sqrt{x}\right)}}\cos\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right).$

**7.** Let $y = \sin(\sin x)$

$\therefore \frac{dy}{dx} = \cos(\sin x)\cos x.$

**8.** Let $y = \sin\left(\sin\sqrt{x}\right)$

$\therefore \frac{dy}{dx} = \cos\left(\sin\sqrt{x}\right)\cos\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right).$

**9.** Let $y = \sin(\log x)$

$\therefore \frac{dy}{dx} = \cos(\log x)\frac{1}{x} = \frac{\cos(\log x)}{x}.$

**10.** Let $y = \log(\sin x)$

$\therefore \frac{dy}{dx} = \left(\frac{1}{\sin x}\right)\cos x = \cot x.$

**11.** Let $y = e^{\sin x}$

$\therefore \frac{dy}{dx} = e^{\sin x}\cos x.$

**12.** Let $y = \sin e^x$

$\therefore \frac{dy}{dx} = \cos e^x\ (e^x) = e^x\cos e^x.$

**13.** Let $y = \sin x^\circ = \sin\left(\frac{\pi x}{180}\right)$

$\therefore \frac{dy}{dx} = \cos\left(\frac{\pi x}{180}\right)\left(\frac{\pi}{180}\right)$

$$= \left(\frac{\pi}{180}\right) \cos x^{\circ}$$

**Note**

A function $f(x)$ is called an odd function if $f(x) = -f(x)$ and $f(x)$ is called an even function if $f(-x) = f(x)$. We can prove that the derivative of an even function is an odd function and the derivative of an odd function is an even function.

## 1.8 Differentiation of inverse function

**Theorem 1.26**

Let $f$ be a continuous one-one function defined on an interval and let $f$ be differentiable at $x$ and $f'(x) \neq 0$. Let $g$ be the inverse of the function $f$. Then $g$ is differentiable at $f(x)$ and $g'\big(f(x)\big) = \frac{1}{f'(x)}$.

**Proof**

Let $f(x) = y$. (1)

$\therefore$ By the definition of inverse function $g(y) = x$ (2)

Let $y + k$ be any point in the domain of $g$.

Since $f$ is 1-1 there exists a unique point say, $x + h$ different from $x$ such that

$f(x + h) = y + k$ (3)

$\therefore g(y + k) = x + h$ (4)

$\therefore \frac{g(y+k)-g(y)}{k} = \frac{(x+h)-x}{f(x+h)-f(x)}$ (by 1, 2, 3, 4 )

$= \frac{1}{\left(\frac{f(x+h)-f(x)}{h}\right)}$ (5)

Since $f$ and g are continuous, as $k \to 0$, we have $h \to 0$. Now, taking limit as $h \to 0$ in (5) and using the fact that $f'(x) \neq 0$ we get $g'(y) = \frac{1}{f'(x)}$

$\therefore g'\big(f(x)\big) = \frac{1}{f'(x)}$.

**Result 17**

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ for all $x \in (1, -1)$.

**Proof**

$\sin x$ is a 1-1 map from $\left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$ onto $[-1, 1]$.

$\therefore \sin^{-1} x$ is defined on $[-1, 1]$.

By the definition of inverse function $y = \sin^{-1} x \Leftrightarrow x = \sin y$.

Let $x \in (1, -1)$. Hence $y \in \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$

Now $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\frac{d}{dy}(\sin y)}$

$$= \frac{1}{\cos y}$$

$$= \frac{1}{\pm\sqrt{(1-\sin^2 y)}}$$

$$= \frac{1}{\sqrt{(1-x^2)}} \qquad \left(\because \cos y > 0 \; in \; \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)\right)$$

Similarly we can derive the following results.

**Result 18:** $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$ for all $x \in (-1, 1)$

**Result 19:** $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ for all $x \in \boldsymbol{R}$.

**Result 20:** $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$ for all $x \in \boldsymbol{R}$.

**Result 21:** $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{(x^2-1)}}$ for all $x \in (-\infty, -1) \cup (1, \infty)$.

**Result 22:** $\frac{d}{dx}(\mathrm{cosec}^{-1} x) = -\frac{1}{|x|\sqrt{(x^2-1)}}$ for all $x \in (-\infty, -1) \cup (1, \infty)$.

**Result 23:** $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{(1+x^2)}}$ for all $x \in \boldsymbol{R}$.

**Result 24:** $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{(x^2-1)}}$ for all $x \in (-1, 1)$.

**Result 25:** $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$ for all $x \in (-1, 1)$.

**Result 26:** $\frac{d}{dx}(\mathrm{sech}^{-1} x) = \frac{1}{x\sqrt{(1-x^2)}}$ for all $x \in (-1, 0) \cup (0, 1)$.

**Result 27:** $\frac{d}{dx}(\mathrm{cosech}^{-1} x) = \frac{1}{|x|\sqrt{(x^2+1)}}$ for all $x \in \boldsymbol{R} - \{\boldsymbol{0}\}$.

**Example 27**

Find the derivatives of the following w.r.t $x$.

1. $\sin^{-1}(2x)$     2. $\sin^{-1}(\sqrt{x})$     3. $(\sin^{-1} x)^2$

4. $\sin^{-1}(x^2)$     5. $\sqrt{(\sin^{-1} x)}$     6. $\sin^{-1}\frac{1}{x}$

7. $\sin^{-1}(e^x)$ 8. $\sin^{-1}(\log x)$ 9. $\sin^{-1}(\sinh^{-1} x)$

**Solution**

**1.** Let $y = \sin^{-1}(2x)$

$\therefore \frac{dy}{dx} = \left(\frac{1}{\sqrt{1-(2x)^2}}\right) 2 = \frac{2}{\sqrt{(1-4x^2)}}.$

**2.** Let $y = \sin^{-1}\left(\sqrt{x}\right)$

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-(2x)^2}}\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{(x-x^2)}}$

**3.** Let $y = (\sin^{-1} x)^2$

$\therefore \frac{dy}{dx} = \frac{2\sin^{-1} x}{\sqrt{(1-x^2)}}$

**4.** Let $y = \sin^{-1}(x^2)$

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{[1-(x^2)^2]}}(2x) = \frac{2x}{\sqrt{(1-x^4)}}$

**5.** Let $y = \sqrt{(\sin^{-1} x)}$

$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{(\sin^{-1} x)}} \frac{1}{\sqrt{(1-x^2)}}$

**6.** Let $y = \sin^{-1} \frac{1}{x}$

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{\left[1-\left(\frac{1}{x^2}\right)\right]}}\left(-\frac{1}{x^2}\right) = -\frac{1}{x\sqrt{(x^2-1)}}$

**7.** Let $y = \sin^{-1}(e^x)$

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{[1-(e^x)^2]}}(e^x) = \frac{e^x}{\sqrt{(1-e^{2x})}}$

**8.** Let $y = \sin^{-1}(\log x)$

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{[1-(\log x)^2]}}\left(\frac{1}{x}\right)$

**9.** Let $y = \sin^{-1}(\sinh^{-1} x)$

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{[1-(\sinh^{-1} x)^2]}} \frac{1}{\sqrt{(1+x^2)}}.$

**Example 28**

If $y = e^{a\sin^{-1} x}$ prove that $(1-x^2)y_1^2 = a^2y^2$.

**Solution**

$y = e^{a\sin^{-1} x}$

$\therefore y_1 = \left(e^{a\sin^{-1} x}\right)\frac{a}{\sqrt{(1-x^2)}}$

$\therefore y_1\sqrt{(1-x^2)} = a\, e^{a\sin^{-1} x} = ay.$

$\therefore (1-x^2)y_1^2 = a^2y^2.$

## 1.9 Differentiation of transformations

Sometimes a function can be simplified by suitable substitution and hence the differentiation becomes easier.

**Example 29**

Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ w. r. t. $x$.

**Solution**

Let $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan\theta$

Then $y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$= \sin^{-1}(\sin 2\theta)$

$= 2\theta$

$= 2\tan^{-1} x$

$\therefore \frac{dy}{dx} = \frac{2}{1+x^2}$

**Example 30**

$y = \tan^{-1}\left(\frac{2x}{1-x^2}\right) + \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) - \tan^{-1}\left(\frac{4x-4x^3}{1-6x^2+x^4}\right)$ show that $\frac{dy}{dx} = \frac{1}{1+x^2}$

**Solution**

Put $x = \tan\theta$

Then, $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}(\tan 2\theta) = 2\theta.$

$\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = \tan^{-1}(\tan 3\theta) = 3\theta.$

$\tan^{-1}\left(\frac{4x-4x^3}{1-6x^2+x^4}\right) = \tan^{-1}(\tan 4\theta) = 4\theta.$

$\therefore y = 2\theta + 3\theta - 4\theta = \theta.$

$= \tan^{-1} x.$

$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}.$

## 1.10 Logarithmic differentiation

When a function is a product of a number of factors it is convenient to take logarithm before differentiation. Also when a function is of the form $u^v$ where $u$ and $v$ are both functions of $x$ it is necessary to take logarithm and then differentiate. This process is known as logarithmic differentiation.

**Example 31**

Find the derivative of $u^v$ where $u$ and $v$ are functions of $x$.

**Solution**

Let $y = u^v$.

$\therefore \log y = v \log u$.

Differentiating w. r. t. $x$ we get $\frac{1}{y}\frac{dy}{dx} = \frac{u}{v}\frac{du}{dx} + \log u \frac{dv}{dx}$.

$\therefore \frac{dy}{dx} = u^v \left[\frac{v}{u}\, u' + v' \log u\right]$.

**Example 32**

Find $y'$ if $y = x^{\sin x}$

**Solution**

$\log y = \sin x \log x$.

$\therefore \frac{1}{y}\frac{dy}{dx} = \sin x \left(\frac{1}{x}\right) + \log x \cos x$.

$\therefore \frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \log x \cos x\right]$.

**Example 33**

Find $y'$ if $y = x^x + x^{\frac{1}{x}}$.

**Solution**

Let $y = u + v$ where $u = x^x$ and $v = x^{\frac{1}{x}}$.

$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$.

Now, $u = x^x$.

$\therefore \log u = x \log x$.

$\therefore \frac{1}{u}\frac{du}{dx} = x\left(\frac{1}{x}\right) + \log x$.

$\therefore \frac{du}{dx} = x^x(1 + \log x)$.

Also $v = x^{\frac{1}{x}}$ .

$\therefore \log v = \left(\frac{1}{x}\right) \log x$.

$\therefore \frac{1}{v}\frac{dv}{dx} = \frac{1}{x}\left(\frac{1}{x}\right) + \log x \left(-\frac{1}{x}\right)$.

$\therefore \frac{dv}{dx} = x^{\frac{1}{x}}\left(\frac{1-\log x}{x^2}\right)$

$\therefore \frac{dy}{dx} = x^x(1 + \log x) + x^{\frac{1}{x}}\left(\frac{1-\log x}{x^2}\right)$.

**Example 34**

If $y = x^{x^x}$ prove $\frac{dy}{dx} = yx^x\left(\frac{1}{x} + \log x + \log^2 x\right)$.

**Solution**

Let $y = x^{x^x}$.

$\therefore \log y = x^x \log x$ (1)

$\therefore \log(\log y) = x \log x + \log(\log \mathrm{x})$.

$\therefore \frac{1}{\log y}\frac{1}{y}\frac{dy}{dx} = \frac{x}{x} + \log x + \frac{1}{\log x}\left(\frac{1}{x}\right)$.

$\therefore \frac{dy}{dx} = y \log y\left(1 + \log x + \frac{1}{x \log x}\right)$.

$= yx^x \log x\left(1 + \log x + \frac{1}{x \log x}\right)$. ( by (1))

$= yx^x\left(\frac{1}{x} + \log x + \log^2 x\right)$.

**Example 35**

If $x^p y^q = (x + y)^{p+q}$ prove that $x\frac{dy}{dx} = y$.

**Solution**

$x^p y^q = (x + y)^{p+q}$.

$\therefore p \log x + q \log y = (p + q)\log(x + y)$.

Differentiating w. r. t. $x$. we get

$\frac{p}{x} + \frac{q}{y}\left(\frac{dv}{dx}\right) = \left(\frac{p+q}{x+y}\right)\left(1 + \frac{dy}{dx}\right)$.

$\therefore \frac{dy}{dx}\left(\frac{q}{y} - \frac{p+q}{x+y}\right) = \frac{p+q}{x+y} - \frac{p}{x}$.

$\therefore \frac{dy}{dx}\left[\frac{qx+qy-py-qv}{y(x+y)}\right] = \frac{px+qx-px-py}{x(x+y)}$

$\therefore \frac{dy}{dx} = \left(\frac{y}{x}\right)\left[\frac{px+qx-px-py}{qx+qy-py-qy}\right] = \left(\frac{y}{x}\right)\left[\frac{qx-py}{qx-py}\right]$.

$= \frac{y}{x}$.

$\therefore x\frac{dy}{dx} = y$.

## 1.11 Parametric Differentiation

**Differentiation of functions represented in terms of a parameter**

Let $x = f(t)$ and $y = g(t)$ where $t$ is a parameter.

Then by chain rule.

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{d(g(t))}{dt} \div \frac{d(f(t))}{dt}$.

**Example 36**

Find $y'$ if $x = a\cos^3 t\,, y = a\sin^3 t$.

**Solution**

$$\frac{dy}{dx} = \frac{\frac{d}{dt}(a\sin^3 t)}{\frac{d}{dt}(a\cos^3 t)}$$
$$= \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t}.$$
$$= -\tan t.$$

**Example 37**

If $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$ prove that $y' = \tan\left(\frac{\theta}{2}\right)$.

**Solution**

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}[a(1-\cos\theta)]}{\frac{d}{d\theta}[a(\theta+\sin\theta)]}.$$
$$= \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)}.$$
$$= \tan\left(\frac{\theta}{2}\right).$$

**1.12 Differentiation of a function with respect to another function**

Let $y = f(x)$ and $z = g(x)$ be two functions both of which have derivatives. To find the differential coefficient of $f$ w. r. t $g$. We treat $x$ as a parameter and we have

$$\frac{dy}{dz} = \frac{df}{dg} = \frac{\left(\frac{df}{dx}\right)}{\left(\frac{dg}{dx}\right)}$$

**Example 38**

Differentiate $e^x$ w. r. t. $\log x$.

**Solution**

Let $y = e^x$ and $z = \log x$.

$$\therefore \frac{dy}{dz} = \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(\log x)}$$
$$= \frac{e^x}{\frac{1}{x}}.$$
$$= x\,e^x.$$

**Example 39**

Find the derivative of $x^{\sin x}$ w. r. t. $(\sin x)^x$.

**Solution**

Let $f = x^{\sin x}$ and $g = (\sin x)^x$.

$\therefore \frac{df}{dg} = \left(\frac{\frac{df}{dx}}{\frac{dg}{dx}}\right)$.

Now, $\log f = \sin x \log x$.

$\therefore \frac{1}{f}\left(\frac{df}{dx}\right) = \frac{\sin x}{x} + \cos x \log x$.

$\therefore \frac{df}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right]$.

Now, $g = (\sin x)^x$.

$\therefore \log g = x \log \sin x$.

$\therefore \frac{1}{g}\frac{dg}{dx} = x \cot x + \log \sin x$.

$\therefore \frac{dg}{dx} = (\sin x)^x[x \cot x + \log \sin x]$.

$\therefore \frac{df}{dg} = \frac{x^{\sin x}}{(\sin x)^x}\left[\frac{\sin x + x \cos x \log x}{x(x \cot x + \log \sin x)}\right]$.

### 1.13 Differentiation of implicit functions

So far we have considered differentiation of functions in which the dependent variable $y$ is expressed explicitly in terms of the independent variable $x$. In this section we shall consider functions of two variables $x$ and $y$ given by $f(x, y) = 0$. We differentiate the function itself and find $\frac{dy}{dx}$.

**Example 40**

Find $y'$ if $x^3 + y^3 = 3\,axy$.

**Solution**

$x^3 + y^3 = 3\,axy$.

Differentiating both sides w. r. t. $x$. We get

$3x^2 + 3y^2y' = 3a(xy' + y)$.

$\therefore y'(3y^2 - 3ax) = 3ay - 3x^2$.

$\therefore y' = \frac{ay - x^2}{y^2 - ax}$.

**Example 41**

If $x^y = e^{x-y}$ prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

**Solution**

$x^y = e^{x-y}$.

$\therefore y \log x = x - y$.

$\therefore y(1 + \log x) = x$.

$\therefore y = \frac{x}{1+\log x}$.

$\therefore \frac{dy}{dx} = \frac{(1+\log x)-x\left(\frac{1}{x}\right)}{(1+\log x)^2}$.

$= \frac{\log x}{(1+\log x)^2}$.

**Example 42**

Find $y'$ if $y = (\sin x)^y$.

**Solution**

$y = (\sin x)^y$.

$\therefore \log y = y \log \sin x$.

$\frac{1}{y}\left(\frac{dy}{dx}\right) = y\left(\frac{\cos x}{\sin x}\right) + (\log \sin x)\left(\frac{dy}{dx}\right)$.

$\therefore \frac{dy}{dx} = \frac{y^2 \cot x}{1-y \log \sin x}$.

**Example 43**

If $x^y = y^x$ prove that $\frac{dy}{dx} = \frac{y(y-x\log y)}{x(x-y\log x)}$.

**Solution**

$x^y = y^x$.

$\therefore y \log x = x \log y$.

Differentiating w. r. t. $x$ on both sides we get

$\frac{y}{x} + \log x\left(\frac{dy}{dx}\right) = \frac{x}{y}\left(\frac{dy}{dx}\right) + \log y$.

$\therefore \frac{dy}{dx}\left(\log x - \frac{x}{y}\right) = \log y - \left(\frac{y}{x}\right)$.

$\therefore \frac{dy}{dx} = \frac{y(x\log y-y)}{x(y\log x-x)}$.

$= \frac{y(y-x\log y)}{x(x-y\log x)}$.

## 1.14 Higher derivatives

Let $y = f(x)$ be a function which is differentiable over an interval $l$. Then its derivative $y_1$ is also a function $x$.

If $y_1$ is differentiable w. r. t. $x$ then its derivative is called the second derivative of $y$ w. r. t. $x$ and is denoted by $\frac{d^2y}{dx^2}$ or $y^{(2)}$ or $y_2$. Generally if we can successively differentiate the function $y = f(x)$ w. r. t. $x$ $n$ times, then the result is denoted by $\frac{d^ny}{dx^n}$ or $y^{(n)}$ or $y_n$ and it is called the n[th] derivative of $y$ w. r. t. $x$

**Example 44**

If $y = x^2 \sin ax$ find $y_1$ and $y_2$.

**Solution**

$y = x^2 \sin ax.$

$\therefore y_1 = x^2(a \cos ax) + 2x \sin ax.$

$\quad = ax^2 \cos ax + 2x \sin ax.$

$\therefore y_2 = ax^2(-a \sin ax) + 2\, ax \cos ax + 2x(a \cos ax) + 2 \sin ax.$

$\quad = (2 - a^2x^2) \sin ax + 4\, ax \cos ax.$

**Example 45**

If $y = e^{-x} \cos x$ prove that $y_4 + 4\, y = 0$.

**Solution**

$y = e^{-x} \cos x$

$\therefore y_1 = -e^{-x} \cos x + e^{-x}(-\sin x) = -e^{-x}(\cos x + \sin x)$

$\therefore y_2 = -[e^{-x}(-\sin x + \cos x) - (\cos x + \sin x)e^{-x}]$

$\quad = -e^{-x}(-2 \sin x) = 2e^{-x} \sin x$

$\therefore y_3 = 2[e^{-x} \cos x - e^{-x} \sin x] = 2e^{-x}(\cos x - \sin x)$

$\therefore y_4 = 2[-e^{-x}(\cos x - \sin x) + e^{-x}(-\sin x - \cos x)]$

$\quad = -2e^{-x}(2 \cos x) = -4e^{-x} \cos x = -4y$

$\therefore y_4 + 4\, y = 0.$

**Example 46**

If $y = e^{a \sin^{-1} x}$ prove $(1 - x^2)y_2 - xy_1 - a^2y = 0$.

**Solution**

$y = e^{a \sin^{-1} x}.$

$\therefore y_1 = \frac{ae^{a \sin^{-1} x}}{\sqrt{(1-x^2)}}$.

$\therefore \sqrt{(1-x^2)}\, y_1 = a\, y$.

$\therefore (1-x^2)\, y_1^2 = a^2\, y^2$.

Now, differentiating again w. r. t. $x$, we get

$(1-x^2)\, 2y_1y_2 - 2xy_1^2 = 2a^2\, y\, y_1$.

$\therefore (1-x^2)y_2 - xy_1 - a^2y = 0$.

**Example 47**

If $y = \left[x + \sqrt{(1+x^2)}\right]^m$, prove that $(1-x^2)y_2 + xy_1 - m^2y = 0$.

**Solution**

$y = \left[x + \sqrt{(1+x^2)}\right]^m$.

$\therefore y_1 = m\left[x + \sqrt{(1+x^2)}\right]^{m-1}\left(1 + \frac{2x}{2\sqrt{(1+x^2)}}\right)$.

$= m\left[x + \sqrt{(1+x^2)}\right]^{m-1}\left(\frac{2\sqrt{(1+x^2)}+2x}{2\sqrt{(1+x^2)}}\right)$.

$= m\left[x + \sqrt{(1+x^2)}\right]^{m-1}\left(\frac{\sqrt{(1+x^2)}+x}{\sqrt{(1+x^2)}}\right)$.

$\therefore y_1\sqrt{(1+x^2)} = m\left[x + \sqrt{(1+x^2)}\right]^m$.

$\therefore (1+x^2)\, y_1^2 = m^2y^2$.

Now, we differentiating again w. r. t. $x$ we get

$(1+x^2)2\, y_1y_2 + 2xy_1^2 = 2m^2yy_1$.

$\therefore (1-x^2)y_2 + xy_1 - m^2y = 0$.

**Exercise 5**

1. If $y = \log\sqrt{\left(\frac{1+\sin x}{1-\sin x}\right)}$ prove that $\frac{dy}{dx} = \sec x$.
2. Prove that the function $y = x\, e^x$ satisfies the equation $xy' = (1-x)y$.
3. Differentiate the following functions w. r. t. $x$.

   (i) $e^{a \sin bx}$ (ii) $e^{\frac{x}{a}} + e^{-\frac{x}{a}}$ (iii) $\sin^m x \cos^n x$

   (iv) $\log(\log(\log x))$
4. If $f(x) = \sqrt{(1+x)}$ find $f(3) + (x-3)f'(3)$.
5. If $f(x) = \tan x$ and $g(x) = \log(1-x)$ find $\left(\frac{f'(0)}{g'(0)}\right)$.

6. Prove that the function $y = e^{-\frac{1}{2}x^2}$ satisfies the equation $xy' = (1 - x^2)y$.

7. If $y = \log\left(\frac{\sqrt{(x^2+1)}-x}{\sqrt{(x^2-1)}+x}\right)$ prove that $\frac{dy}{dx} = -\frac{2}{\sqrt{(1+x^2)}}$.

8. Differentiate the functions w. r. t. $x$.

(i) $\cot^{-1}(\log x)$ (ii) $\tan^{-1}\left(\frac{\sqrt{(1+x^2)}-1}{x}\right)$

(iii) $\tan^{-1}\frac{(1+x)}{(1-x)}$ (iv) $\cot^{-1}(\sinh x)$

9. Find $\frac{dy}{dx}$ if $y = \tan^{-1}\left\{\frac{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}\right\}$

10. Differentiate the following w. r. t. $x$.

(i) $\sin^{-1}(3x - 4x)$ (ii) $\sin^{-1}\left[2x\sqrt{(1-x^2)}\right]$

(iii) $\tan^{-1}\left(\frac{1+\cos x}{1-\cos x}\right)^{\frac{1}{2}}$

11. Differentiate $(2x)^x$ w. r. t. $x$.

12. Differentiate the functions w. r. t. $x$.

(i) $2^x$ (ii) $e^{x^x}$ (iii) $x^{\sin^{-1}x}$ (iv) $x^x + (\cot x)^x$

(v) $x^{\sqrt{x}}$ (vi) $a^{x^2}$ (vii) $\left(\frac{1}{x}\right)^{\frac{1}{x}}$

13. If $y = a^{xy}$ prove that $\frac{dy}{dx} = \frac{y^2 \log a}{1-\log y}$.

14. Find $\frac{dy}{dx}$ if

(i) $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$

(ii) $y = (\sin x)^{\tan x} + (\tan x)^{\sin x}$

(iii) $y = x^x + a^x$ (iv) $y = x^{\tan y} + (\sin x)^{\cos x}$

15. If $x^{1+x} + y^{1+x} = a$ prove that $\frac{dy}{dx} = -\frac{y[-xy^{1+x}\log y + x^{1+x}(1+y)]}{[y\,x^{1+x}x\log x + y^{1+x}(1+x)]}$.

16. If $y = x^{x^y}$ prove $\frac{dy}{dx} = \frac{y^2 \log y}{x(1-y\log x \log y)^2}$

17. If $y = (\cos x)^y$ prove $\frac{dy}{dx} = -\frac{y^2 \tan x}{1-y\log\cos x}$

18. Find $y'$ for $(\sin x)^{\cos y} = (\cos y)^{\sin x}$.

19. If $y = x^y$ prove that $x\frac{dy}{dx} = \frac{y^2}{1-y\log x}$.

20. $y = \sqrt{(\sin x + y)}$ prove that $\frac{dy}{dx} = \frac{\cos x}{2y-1}$

21. If $y = (\sin x)^y$ prove that $\frac{dy}{dx} = \frac{y^2 \cot x}{1-y\log\sin x}$

22. If $(a+bx)e^{\frac{y}{x}} = x$ prove that $x^3y_2 = (xy_1 - y)^2$.

23. If $y = x^2e^{ax}$, prove that $y_2 = e^{ax}(a^2x^3 + 6ax^2 + 6x)$

24. If $y = (\sin^{-1} x)^2$ prove that $(1-x^2)y_2 - xy_1 + 2$.

25. If $y = a\cos(\log x) + b\sin(\log x)$ prove that $x^2y_2 + xy^1 + y = 0$.

26. If $y = \tan(m\tan^{-1} x)$ prove that $(1+x^2)y_1 = m(1+y^2)$.

27. If $y = \frac{\log x}{x}$ prove that $y_2 = \frac{2\log x - 3}{x^3}$

28. If $y = (\tan^{-1} x)^2$ prove $(1+x^2)^2y_2 + 2x(1+x^2)y_1 = 2$.

**Answers**

3. (i) $ab\, e^{a\sin bx}$ (ii) $\frac{e^{x/a} - e^{-x/a}}{a}$

(iii) $\sin^{m-1} x \cos^{n-1} x\,(m\cos^2 x - n\sin^2 x)$ (iv) $\frac{1}{\log(\log x)x\log x}$

4. $\frac{x}{4} + \frac{5}{4}$ 5. $-1$.

8. (i) $-\frac{1}{x(1+\log^2 x)}$ (ii) $\frac{1}{2(1+x^2)}$ (iii) $\frac{1}{1+x^2}$ (iv) – sech $x$

9. $\frac{x}{\sqrt{1-x^4}}$

10. (i) $\frac{3}{\sqrt{1-x^2}}$ (ii) $\frac{2}{\sqrt{1-x^2}}$ (iii) $-\frac{1}{2}$

11. $(2x)^x[1 + \log 2x]$

12. (i) $2x\log 2$ (ii) $e^{x^x}\log e^{x^x}[1 + \log x]$

(iii) $x\sin^{-1} x\left[\frac{\sin^{-1} x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right]$

(iv) $x^x[1 + \log x] + (\cot x)^x[\log\cot x - 2x\,\mathrm{cosec}\, x]$

(v) $x^{\sqrt{x}-\frac{1}{2}} - [1 + \log x]$ (vi) $a^{x^2}[2x\log a]$

(vii) $-\left(\frac{1}{x}\right)^{\frac{1}{x}}\left(\frac{1}{x}\right)\left[1 + \log\left(\frac{1}{x}\right)\right]$

14.(i) $(\sin x)^{\cos x}[\cos x\cot x - \sin x\log\sin x]$

$+(\cos x)^{\sin x}[\cos x\log(\cos x) - \sin x\tan x]$

(ii) $(\sin x)^{\tan x}[1 + \sec^2 x\log(\sin x)]$

$+(\tan x)^{\sin x}[\sec x + \cos x\log x\log(\tan x)]$

(iii) $x^x[1 + \log x] + a^x\log a$

(iv) $\frac{y}{x}\left[\frac{\tan y + x\cot x\cos x - x\sin x\log\sin x}{1 - y\sec^2 y\log x}\right]$

18. $\left[\frac{\cos x\log\cos y - \cos y\cot x}{\sin x\tan y - \sin y\log\sin x}\right]$.

## 1.15 $n^{th}$ Derivative and Leibnitz theorem

### 1.15.1 $n^{th}$ Derivative of some standard functions

**Theorem 1.27**

$\frac{d^n}{dx^n}(ax+b)^m = m(m-1)(m-2)\dots(m+n-1)a^n(ax+b)^{m-n}$

**Proof**

Let $y = ax + b$.

$\therefore y_1 = m(ax+b)^{m-1}a = am(ax+b)^{m-1}$.

$y_2 = m(m-1)a^2(ax+b)^{m-2}$

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……………………………………..

$y_n = m(m-1)(m-2)\dots(m+n-1)a^n(ax+b)^{m-n}$ .

**Theorem 1.28**

$\frac{d^n}{dx^n}(ax+b)^n = n!\,a^n$.

**Proof**

Put $m = n$ in Theorem 1.27 to get the result.

**Theorem 1.29**

$\frac{d^n}{dx^n}\left(\frac{1}{ax+b}\right) = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$

**Proof**

Put $m = -1$ in Theorem 1.27 to get the result.

**Theorem 1.30**

$\frac{d^n}{dx^n}[\log(ax+b)] = \frac{(-1)^{n-1}(n-1)! a^n}{(ax+b)^n}$.

**Proof**

Let $y = \log(ax+b)$

$\therefore y_1 = a(ax+b)^{-1}$.

From Theorem 1.29 we get $y_n = \frac{(-1)^{n-1}(n-1)! a^n}{(ax+b)^n}$.

**Theorem 1.31**

$\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$.

**Proof**

Let $y = e^{ax}$.

$\therefore y_1 = a\,e^{ax}$
$y_2 = a^2 e^{ax}$
……………..
……………..
$y_n = a^n e^{ax}.$

**Theorem 1.32**

$\frac{d^n}{dx^n}[\sin(ax+b)] = a^n \sin\left(ax + b + \frac{1}{2}(n\pi)\right).$

**Proof**

Let $y = \sin(ax+b)$

$\therefore y_1 = a\cos(ax+b) = a\sin\left(ax + b + \frac{1}{2}\pi\right).$

$y_2 = a^2\cos\left(ax + b + \frac{1}{2}\pi\right) = a^2\sin\left(ax + b + \frac{1}{2}(2\pi)\right)$

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$y_n = a^n\sin\left(ax + b + \frac{1}{2}(n\pi)\right).$

**Theorem 1.33**

$\frac{d^n}{dx^n}[\cos(ax+b)] = a^n \cos\left(ax + b + \frac{1}{2}(n\pi)\right).$

**Proof**

Proof is similar to the Theorem 1.32

**Theorem 1.34**

$\frac{d^n}{dx^n}[e^{ax}\sin(bx+c)] = r^n e^{ax}\sin(bx + c + n\theta)$ where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}(b/a)$.

**Proof**

Let $y = e^{ax}\sin(bx+c)$.
$\therefore y_1 = a\,e^{ax}\sin(bx+c) + b\,e^{ax}\cos(bx+c)$
Put $a = r\cos\theta$ and $b = r\sin\theta$.
$\therefore r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}(b/a)$.
Also $y_1 = r\,e^{ax}[\sin(bx+c)\cos\theta + \cos(bx+c)\sin\theta]$
$\quad = r\,e^{ax}\sin(bx+c+\theta)$
Similarly $y_2 = r^2 e^{ax}\sin(bx+c+2\theta)$

……………………………………

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$y_n = r^n e^{ax}\sin(bx + c + n\theta)$.

$\therefore y_n = \left[\sqrt{a^2+b^2}\right]^n \sin[bx + c + n\tan^{-1}(b/a)]$.

**Theorem 1.35**

$\frac{d^n}{dx^n}[e^{ax}\cos(bx+c)] = r^n e^{ax}\cos[bx + c + n\tan^{-1}(b/a)]$ where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}(b/a)$.

**Proof**

Proof is similar to that of Theorem 1.34

**Example 48**

Find $y_n$ if $y = \frac{3x^2-1}{(x-1)^2(2x+1)}$

**Solution**

$y = \frac{3x^2-1}{(x-1)^2(2x+1)}$

Splitting into partial fractions we get

$y = \frac{14}{9}\left(\frac{1}{x-1}\right) + \frac{2}{3}\left(\frac{1}{(x-1)^2}\right) - \frac{1}{9}\left(\frac{1}{(2x+1)}\right)$

$\therefore y_n = \frac{14}{9}\left[\frac{(-1)^n n!}{(x-1)^{n+1}}\right] + \frac{2}{3}\left[\frac{(-1)^n (n+1)!}{(x-1)^{n+2}}\right] - \frac{1}{9}\left[\frac{(-1)^n 2^n n!}{(2x+1)^{n+1}}\right]$.

**Example 49**

Find $y_n$ if $y = \sin 3x \cos x$.

**Solution**

$y = \sin 3x \cos x$.

$= \frac{1}{2}(\sin 4x + \sin 2x)$.

$\therefore y_n = \frac{1}{2}\left[4^n \sin\left(4x + \frac{1}{2}n\pi\right) + 2^n \sin\left(2x + \frac{1}{2}n\pi\right)\right]$.

**Example 50**

Find $y_n$ if $y = \log\frac{2x+3}{3x+2}$

**Solution**

$y = \log\frac{2x+3}{3x+2} = \log(2x+3) - \log(3x+2)$

$\therefore y_n = \frac{(-1)^{n-1}(n-1)!2^n}{(2x+3)^n} - \frac{(-1)^{n-1}(n-1)!3^n}{(3x+2)^n}$

$$= (-1)^{n-1}(n-1)!\left[\frac{2^n}{(2x+3)^n} - \frac{3^n}{(3x+2)^n}\right].$$

**Exercise 6**

Find the n^th differential coefficient of the following:

1. $\frac{3x}{(2x+1)(x-1)}$ 2. $\frac{x^2}{(x-2)(x-1)^3}$ 3. $\cos^2 x$

4. $\sin^3 x$ 5. $\sin x \sin 2x \sin 3x$ 6. $\cos^7 x \sin^3 x$

7. $\log(4 - x^2)$ 8. $e^{ax}\, cos^2\, bx$ 9. $e^{2x}\cos 4x$

**Answers**

1. $(-1)^n n!\left[\frac{1}{(x-1)^{n+1}} + \frac{2^n}{(2x+1)^{n+1}}\right]$

2. $(-1)^{n-1}n!\frac{1}{2}(n+1)(n+2)(x-1)^{-n-3} + 3(n+1)(x-1)^{-n-2} + 4(x-1)^{-n-1} - 4(x-2)^{-n-1}$

3. $2^{n-1}\cos\left(2x + \frac{1}{2}n\pi\right)$

4. $\frac{3}{4}\sin\left(x + \frac{1}{2}n\pi\right) - \frac{3^n}{4}\sin\left(3x + \frac{1}{2}n\pi\right)$

5. $\frac{1}{4}\left[2^n\sin\left(2x + \frac{1}{2}n\pi\right) + 4^n\sin\left(4x + \frac{1}{2}n\pi\right) - 6^n\sin\left(6x + \frac{1}{2}n\pi\right)\right]$

6. $-\frac{\left[10^n\sin\left(10x+\frac{1}{2}n\pi\right) - 4(8^n)\sin\left(8x+\frac{1}{2}n\pi\right) + 14(2^n)\sin\left(2x+\frac{1}{2}n\pi\right)\right]}{2^{-9}}$

7. $(n-1)!\left[(-1)^{n-1}(2+x)^{-n} - (2-x)^{-n}\right]$

8. $\frac{1}{2}\left[a^n e^{ax} + (a^2 + 4b^2)^{\frac{n}{2}}\, e^{ax}\cos 2bx + n\tan^{-1}\left(\frac{2b}{a}\right)\right]$

9. $\sqrt{20}\, e^{2x}\cos(4x + n\tan^{-1} 2)$

### 1.15.2 Leibnitz's Theorem

We now prove Leibnitz's Theorem on nth differential coefficient of the product of two functions.

**Theorem 1.36 (Leibnitz's Theorem)**

If u and v are functions of x possessing derivatives of n^th order, then

$$(uv)_n = uv_n + nC_1u_1v_{n-1} + nC_2u_2v_{n-2} + \cdots + nC_ru_rv_{n-r} + \cdots + nC_nu_nv$$

**Proof**

We prove this theorem by induction on n.

We note that $(uv)_1 = \frac{d}{dx}(u\, v) = uv_1 + u_1v.$

Thus the theorem is true for $n = 1$.

Now, let us assume that the theorem is true for $n = m$.

$\therefore (uv)_m = uv_m + mC_1 u_1 v_{m-1} + \cdots + mC_{r-1} u_{r-1} v_{m-r+1}$
$+ mC_r u_r v_{m-r} + \cdots + mC_m u_m v$

Differentiating both sides with respect to $x$ we get,

$(uv)_{m+1} = uv_{m+1} + u_1 v_m + mC_1[u_1 v_m + u_2 v_{m-1}] + \cdots +$
$mC_{r-1}[u_{r-1} v_{m-r+2} + u_r v_{m-r+1}] + mC_r[u_r v_{m-r+1} +$
$u_{r+1} v_{m-r}] + \cdots + mC_m[u_m v_1 + u_{m+1} v]$
$= uv_{m+1} + u_1 v_m + mC_1 u_1 v_m + mC_1 u_2 v_{m-1} + \cdots$
$+ mC_{r-1} u_{r-1} v_{m-r+2} + mC_{r-1} u_r v_{m-r+1} + mC_r u_r v_{m-r+1}$
$+ mC_r u_{r+1} v_{m-r} + \cdots + mC_m u_m v_1 + mC_m u_{m+1} v$
$= uv_{m+1} + (1 + mC_1) u_1 v_m + [mC_1 + mC_2] u_2 v_{m-1} + \cdots$
$+ [mC_{r-1} + mC_r] u_r v_{m-r+1} + \cdots + mC_m u_{m+1} v.$

Now we can use these equations $mC_m = 1, mC_0 = 1$ and $mC_r + mC_{r-1} = (m+1)C_r$ and reduce the above equation is

$(uv)_{m+1} = uv_{m+1} + (m+1)C_1 u_1 v_m + (m+1)C_2 u_2 v_{m-1} + \cdots$
$+ (m+1)C_r u_r v_{m-r+1} + \cdots + (m+1)C_{m+1} u_{m+1} v$

$\therefore$ The theorem is true for $n = m + 1$.

Hence the theorem is true for all $n \in \boldsymbol{N}$.

**Example 51**

If $y = x^2 e^{ax}$ find $y_n$.

**Solution**

Let $y = uv$ where $u = x^2$ and $v = e^{ax}$.

$\therefore$ By Leibnitz's theorem

$y_n = x^2 (e^{ax})_n + nC_1 (x^2)_1 (e^{ax})_{n-1} + \cdots$
$= x^2 (a^n e^{ax})_n + nC_1 (2x)(a^{n-1} e^{ax}) + nC_2 (2)(a^{n-2} e^{ax})$
$= a^{n-2} e^{ax} [a^2 x^2 + 2anx + n(n-1)].$

**Example 52**

If $y = a\cos(\log x) + b\sin(\log x)$ prove that $x^2 y_2 + x y_1 + y = 0$. Hence prove that $x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2 + 1) y_n = 0$.

**Solution**

$y = a\cos(\log x) + b\sin(\log x)$.

Differentiating w. r. t. $x$ we get $y_1 = -\frac{a\sin(\log x)}{x} + \frac{b\cos(\log x)}{x}$

$\therefore x y_1 = -a\sin(\log x) + b\cos(\log x)$.

Differentiating w. r. t. $x$ again we get

$xy_2 + y_1 = -\frac{a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$

$= -\frac{1}{x}[a\cos(\log x) + b\sin(\log x)]$

$= -\frac{y}{x}.$

$\therefore x^2y_2 + xy_1 + y = 0$

Using Leibnitz's theorem for n$^{th}$ derivative we get

$[x^2y_{n+2} + nC_1(2x)y_{n+1} + nC_2(2)y_n] + [xy_{n+1} + nC_1y_n] + y_n = 0.$

$\therefore x^2y_{n+2} + (2nx + x)y_{n+1} + [n(n-1) + n + 1]y_n = 0.$

$\therefore x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0.$

**Example 53**

If $y = \left(x + \sqrt{1+x^2}\right)^m$ prove that $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0.$

**Solution**

$y = \left(x + \sqrt{1+x^2}\right)^m$

$\therefore y_1 = m\left[x + \sqrt{1+x^2}\right]^{m-1}\left(1 + \frac{2x}{2\sqrt{1+x^2}}\right)$

$\therefore y_1\sqrt{1+x^2} = m\left[x + \sqrt{1+x^2}\right]^m$

$\therefore (1+x^2)\, y_1^2 = m^2y^2.$

Now, we differentiating again w. r. t. $x$ we get

$(1+x^2)2\, y_1y_2 + 2xy_1^2 = 2m^2yy_1.$

$\therefore (1-x^2)y_2 + xy_1 - m^2y = 0.$

Differentiating n times using Leibnitz's theorem we get

$[(1+x^2)y_{n+2} + nC_1(2x) + nC_2y_n(2)] + [xy_{n+1} + nC_1y_n] - m^2y_n = 0.$

$\therefore (1+x^2)y_{n+2} + (2n+1)xy_{n+1} + [n(n-1) + n - m^2]y_n = 0.$

$\therefore (1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0.$

**Exercise 7**

1. Find the nth derivative of the following:

   (i) $x^3e^{ax}$ (ii) $x^ne^x$ (iii) $x^2\cos x$ (iv) $x^3a^x$

2. If $y = e^{a\sin^{-1}x}$ prove that $(1-x^2)y_2 - xy_1 - a^2y = 0$. Hence prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

3. If $y = \cos(\log x)$ prove that $(1-x^2)x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0.$

4. If $y = (1-x)^a e^{ax}$ prove that $y_1(1-x) + axy = 0$. Hence deduce that $(1-x)y_{n+1} - (n+ax)y_n - nay_{n-1} = 0$.
5. If $y = (x^2-1)^n$ prove that $(x^2-1)y_{n+2} - 2xy_{n+1} - n(n+1)y_n = 0$.

**Answers**

1. (i) $a^{n-3}e^{ax}[a^3x^2 + 3na^2x^2 + 3n(n-1)ax + n(n-1)(n-2)]$

(ii) $e^x\left[x^n + \frac{n^2}{1!}x^{n-1} + \frac{n^2(n-1)^2}{2!} + \cdots + \frac{n^2(n-1)^2\ldots1^2}{n!}\right]$

(iii) $x^2\cos\left(x + \frac{1}{2}n\pi\right) + 2n\pi\cos\left[x + \frac{1}{2}(n-1)\pi\right]$

$+n(n-1)\cos\left[x + \frac{1}{2}(n+2)\pi\right]$.

(iv) $a^x(\log a)^{n-2}[x^2(\log a)^2 + 2nx\log a + n(n-1)]$.

## 1.16 Partial Differentiation

In a real valued function of several variables we assign fixed values to all but one of the variables and allow only that variable to vary, then the function virtually becomes a function of one variable.

For example, consider the function $z = f(x,y) = x^2 + 2x + y^2$. If we fix the value 2 to $y$ then $z = f(x,2) = x^2 + 2x + 4$ is a function of the single variable $x$. If we differentiate this function with respect to x at $x = 1$. We obtain the partial derivative of $f(x,y)$ w. r. t. $x$ at $(1,2)$.

**We now give the normal definition of partial derivatives.**

**Definition**

Let $z = f(x,y)$ be a function of two variables. If $\lim_{h\to0}\left[\frac{f(x_1+h,y_1)-f(x_1,y_1)}{h}\right]$ exists and is finite we say that the partial derivative of $f$ w. r. t. $x$ at $(x_1,y_1)$ exists and its value is given by the above limit. We denote this by $\frac{\partial z}{\partial x}$ at $(x_1,y_1)$ or $f_x(x_1,y_1)$ or $D_xf(x_1,y_1)$ or $D_1f(x_1,y_1)$. Thus we have

$$\frac{\partial z}{\partial x} = \lim_{h\to0}\left[\frac{f(x_1+h,y_1)-f(x_1,y_1)}{h}\right]$$

Similarly the partial derivative of $f(x,y)$ w. r. t. $y$ at $(x_1,y_1)$ is defined as

$$\frac{\partial z}{\partial y} = \lim_{k\to0}\left[\frac{f(x_1,y_1+k)-f(x_1,y_1)}{k}\right]$$

If it exists and is finite and it is also denoted by $f_y(x_1,y_1)$ or $D_yf(x_1,y_1)$ or $D_2f(x_1,y_1)$.

**Note**

If we have a function of $n$ independent variables we can define as above, the partial derivative w. r. t. any one of the variables.

If $z = f(x, y)$ posses a partial derivative w. r. t. $x$ at every point of its domain, then we get a new function $\frac{\partial z}{\partial y}$. This function is also a function of $x$ and $y$ which may be differentiated w. r. t. either of the independent variables, thus giving partial derivatives of higher order. We have $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$; $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$; $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$; $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ which are also denoted by $\frac{\partial^2 f}{\partial x^2}$; $\frac{\partial^2 f}{\partial y^2}$; $\frac{\partial^2 f}{\partial y \partial x}$; $\frac{\partial^2 f}{\partial x \partial y}$ or $f_{xx}$; $f_{yy}$; $f_{xy}$; $f_{yx}$ respectively.

Thus we have four second order partial derivatives. The other higher order partial derivatives can similarly be defined.

**Example 54**

If $u = \log(\tan x + \tan y + \tan z)$, show that $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$.

**Solution**

$u = \log(\tan x + \tan y + \tan z)$.

$\therefore \frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$

$\therefore \sin 2x \frac{\partial u}{\partial x} = \frac{2 \tan x}{\tan x + \tan y + \tan z}$

$\therefore \sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = \frac{2 \tan x + 2 \tan y + 2 \tan z}{\tan x + \tan y + \tan z} = 2.$

**Example 55**

If $f = x^3 + y^3 + z^3 + 3xyz$ find (i) $f_x$ (ii) $f_{xx}$ (iii) $f_{xyz}$

**Solution**

$f = x^3 + y^3 + z^3 + 3xyz$

$\therefore f_x = 3x^2 + 3yz$

$\therefore f_{xx} = \frac{\partial}{\partial x}(3x^2 + 3yz) = 6x$

$\therefore f_{xy} = \frac{\partial}{\partial y}(3x^2 + 3yz) = 3z$

$\therefore f_{xyz} = \frac{\partial}{\partial z}(f_{xy}) = \frac{\partial}{\partial z}(3z) = 3$

**Example 56**

If $u = f(x - y, y - z, z - x)$ prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

**Solution**

$u = f(x - y, y - z, z - x).$

Put $X = y - z$ $\quad \therefore \frac{\partial X}{\partial x} = 0; \frac{\partial X}{\partial y} = 1; \frac{\partial X}{\partial z} = -1$ (1)

Put $Y = z - x$ $\quad \therefore \frac{\partial Y}{\partial x} = -1; \frac{\partial Y}{\partial y} = 0; \frac{\partial Y}{\partial z} = 1$ (2)

Put $Z = x - y$ $\quad \therefore \frac{\partial Z}{\partial x} = 1; \frac{\partial Z}{\partial y} = -1; \frac{\partial Z}{\partial z} = 0$ (3)

Now, $u = f(X, Y, Z)$ where $X, Y, Z$ are functions of $x, y$ and $z$.

$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial x}$

$= -\frac{\partial u}{\partial Y} + \frac{\partial u}{\partial Z}$ [ using (1)]

Similarly,

$\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial Z} + \frac{\partial u}{\partial X}$ and

$\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial X} + \frac{\partial u}{\partial Y}.$

$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

**Exercise 8**

1. If $u = x^2yz + xy^2z + xyz^2$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 4xyz(x + y + z).$

2. If $p = q^2r^2$ prove that $\frac{\partial^2 p}{\partial q^2} \times \frac{\partial^2 q}{\partial p^2} = 4p.$

3. If $u = e^{xy}$ prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right].$

4. If $u = \log(x^2 + y^2 + z^2)$ prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{x^2+y^2+z^2}.$

5. If $u = ax + 6y + 8z$ and $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ find $a$.

**Answer**

5. $-14.$

## 1.17 Euler's Theorem

### Homogeneous function and Euler's theorem

For simplicity, definition and theorems in this section will be given for functions of two variables only. Extension to functions of $n$ variables is immediate.

### Definition

Consider the polynomial $f(x,y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_ny^n$. Here the degree of each term is $n$. We say that $f$ is a homogeneous function of degree $n$.

We now extend the notion of homogeneity to functions other than polynomials.

A function $f(x,y)$ is said to be homogeneous of degree $n$ if $f(\lambda x, \lambda y) = \lambda^n f(x,y)$ for all $\lambda$.

### Example 57

$f(x,y) = x^3 + y^3 + 3x^2y$ is a homogeneous function of degree 3.

**Solution**

For, $f(\lambda x, \lambda y) = \lambda^3x^3 + \lambda^3y^3 + 3\lambda^2x^2\lambda y$

$$= \lambda^3(x^3 + y^3 + 3x^2y)$$

$$= \lambda^3 f(x,y).$$

### Example 58

$f(x,y) = (x^2 + 4y^2)^{-\frac{1}{3}}$ is a homogeneous function of degree $-\frac{2}{3}$.

### Example 59

$f(x,y) = \sin\left(\frac{x+y}{x-y}\right)$ is a homogeneous function of degree 0.

Note $f(x,y) = x^2 + x - y$ is not a homogeneous function.

### Theorem 1.37 (Euler's Theorem)

Let $f(x,y)$ be a homogeneous function of degree $n$ having first order partial derivatives in a domain D of $\boldsymbol{R^2}$.

Then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n\, f(x,y)$ for all $(x,y) \in D$.

**Proof**

Since $f(x, y)$ is a homogeneous function of degree $n$ we have $f(\lambda x, \lambda y) = \lambda^n f(x, y)$.

Differentiating both sides w. r. t $\lambda$ we have

$x\, f_x(\lambda x, \lambda y) + y\, f_y(\lambda x, \lambda y) = n\, \lambda^{n-1} f(x, y)$.

Putting $\lambda = 1$ we have $x\, f_x(x, y) + y\, f_y(x, y) = n\, f(x, y)$.

(i.e.) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n\, f(x, y)$.

**Theorem 1.38 (Extension of Euler's Theorem)**

If $f(x, y)$ is a homogeneous function of degree $n$ then $x^2 f_{xx} + 2xy\, f_{xy} + y^2 f_{yy} = n(n-1)f$.

**Proof**

By Euler's theorem we have $x\, f_x + y\, f_y = nf$ (1)

Differentiating (1) w. r. t. $x$ and $y$ we get

$x\, f_{xx} + f_x + y\, f_{yx} = nf_x$ (2)

$x\, f_{xy} + f_y + y\, f_{yy} = nf_y$ (3)

Multiplying (2) by $x$ and (3) by $y$ and adding we get

$x^2 f_{xx} + 2xy\, f_{xy} + y^2 f_{yy} + \left(x\, f_x + y\, f_y\right) = n\left(x\, f_x + y\, f_y\right)$

$\therefore x^2 f_{xx} + 2xy\, f_{xy} + y^2 f_{yy} = n(n-1)f$ [ using (1)]

**Example 60**

Verify Euler's theorem for the function $f = x^3 - 2x^2 y + 3xy^2 + y^3$

**Solution**

Clearly $f$ is a homogeneous function of degree 3.

$f_x = 3x^2 - 4xy + 3y^2$.

$f_y = -2x^2 + 6xy + 3y^2$.

$\therefore xf_x + yf_y = x(3x^2 - 4xy + 3y^2) + y(-2x^2 + 6xy + 3y^2)$.

$= 3(x^3 - 2x^2 y + 3xy^2 + y^3)$

$= 3f$.

Hence Euler's Theorem is verified.

**Example 61**

If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ prove that $xu_x + yu_y = \frac{1}{2}\tan u$.

**Solution**

Let $v = \frac{x+y}{\sqrt{x}+\sqrt{y}}$.

Then $u = \phi^{-1}(v)$ where v is a homogeneous function of degree $\frac{1}{2}$ and $\phi(u) = \sin u$.

∴ By Theorem 2, $xu_x + yu_y = \frac{1}{2}\frac{\phi(u)}{\phi'(u)} = \frac{\sin u}{2\cos u} = \frac{\tan u}{2}$.

**Example 62**

If $f(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2+y^2}$ prove $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f = 0$

**Solution**

$f(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2+y^2}$

∴ $f(\lambda x, \lambda y) = \frac{1}{\lambda^2 x^2} + \frac{1}{\lambda x \lambda y} + \frac{\log \lambda x - \log \lambda y}{\lambda^2 x^2 + \lambda^2 y^2}$

$= \frac{1}{\lambda^2 x^2} + \frac{1}{\lambda x \lambda y} + \frac{(\log \lambda + \log x) - (\log \lambda + \log y)}{\lambda^2 x^2 + \lambda^2 y^2}$

$= \frac{1}{\lambda^2}\left[\frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2+y^2}\right]$

Hence $f(x,y)$ is a homogeneous function of degree $-2$.

∴ By Euler's theorem $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -2f$.

∴ $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f = 0$

**Exercise 9**

1. Verify Euler's Theorem for the following functions.
   (i) $u = x^3 - 2x^2y + y^3$ (ii) $u = ax^2 + 2hxy + by^2$
2. If $u = \sin^{-1}\left[\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right]$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.
3. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.
   Hence or otherwise prove that
   $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2 u)\sin 2u$.
4. If $u = \frac{xy}{x+y}$ prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$.

# UNIT-II

## APPLICATION OF DIFFERENTIATION I

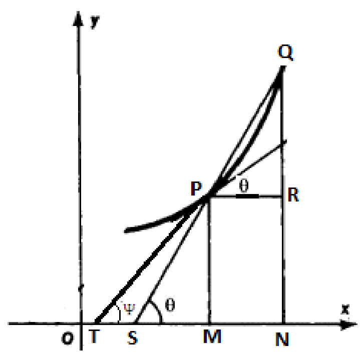
### 2.0 Introduction

In this chapter we shall discuss some important applications of differentiation such as the geometrical applications – tangent, normal sub tangent, subnormal, angle of intersection, radius of curvature, evolutes and envelope of Cartesian and polar curves.

### 2.1 Tangent, Normal, Sub tangent, Subnormal

### 2.1.1 Tangent and Normal

Consider a function $y = f(x)$ which is continuous in $[a, b]$. Suppose $f(x)$ is differentiable at a point $c \in (a, b)$. Let $P$ be the point $\left(c, f(c)\right)$ which lies on the graph of $f$. Let $Q\left(c + h, f(c + h)\right)$ be a neighboring point. Draw $PM$ and $QN$ perpendicular to $x$-axis. From $P$ draw $PR$ perpendicular to $QN$. Then we have $PM = f(c)$ and $QN = f(c + h)$. Let the chord $PQ$ intersect the $x$- axis at $S$ making an angle $\theta$ with the positive direction of the $x$-axis measured in the anticlockwise direction. Let the tangent at $P$ to the curve $y = f(x)$ intersect the $x$-axis at $T$ making an angle $\varphi$ with the positive direction of the $x$-axis measured in the anticlockwise direction.




Now as the point $Q$ approaches $P$ along the graph, both $QR$ and $PR$ tend to zero. Also the chord $PQ$ in the limiting position tends to the tangent at $P$ and the angle $\theta$ tends to angle $\varphi$.

$$\therefore f'(c) = \lim_{h\to 0}\left[\frac{f(c+h)-f(c)}{h}\right]$$
$$= \lim_{h\to 0}\left[\frac{QN-PM}{PR}\right]$$
$$= \lim_{h\to 0}\left[\frac{QR}{PR}\right]$$
$$= \lim_{h\to 0}\tan\theta = \tan\varphi.$$

Hence the derivative $f'(c)$ represents the slope of the tangent to $y = f(x)$ at $(c, f(c))$.

### 2.1.2 Equation of tangent and normal

Since the slope of the tangent at any point $(x_1, y_1)$ on the curve $y = f(x)$ is $f'(x_1)$ we see that the Equation of the tangent to the curve at $(x_1, y_1)$ is given by

$$y - y_1 = f'(x_1)(x - x_1)$$

Since the normal to the curve $y = f(x)$ at $(x_1, y_1)$ is perpendicular to the tangent to the curve at that point the slope of the normal is given by $-\left(\frac{1}{f'(x_1)}\right)$ provided $f'(x_1) \neq 0$.

Hence the equation of normal to the curve at $(x_1, y_1)$ is given by

$$y - y_1 = -\left(\frac{1}{f'(x_1)}\right)(x - x_1)$$

### 2.1.3 Sub tangent, Subnormal, Length of Tangent, Normal, Sub tangent and Subnormal

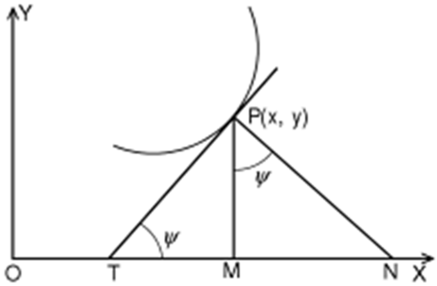
**Definition**

Let $P(x, y)$ be any point on the curve $y = f(x)$. Let the tangent and the normal at $P$ meet the $x$-axis at $T$ and $N$ respectively. Draw $PM$ perpendicular to the $x$-axis. $PT$ and $PN$ are the length of the tangent and normal to the curve and $TM$ and $MN$ are the subtangent and subnormal to the curve at the point $P$.

Let $\psi$ be the angle that the tangent makes with the $x$-axis.

**Length of the tangent** $= PT = y\,\text{cosec}\,\psi$

$$= y\sqrt{1+\cot^2\psi}$$
$$= \frac{y\sqrt{\tan^2\psi+1}}{\tan\psi}$$
$$= \frac{y\sqrt{1+y'^2}}{y'}$$




**Length of normal** $= PN = y \sec\psi$

$$= y\sqrt{1+\tan^2\psi}$$

$$= y\sqrt{1+y'^2}$$

**Length of the subtangent** $= TM = y\cot\psi$

$$= \frac{y}{\tan\psi} = \frac{y}{y'}$$

**Length of the subnormal** $= MN = y\tan\psi$

$$= yy'.$$

**2.1.4 Polar Tangent, Normal, Sub-tangent, Sub-normal**

Let $P(r,\theta)$ be a point on the polar curve whose equation is $r = f(\theta)$. Draw the tangent and the normal to the curve at $P$. Through the pole $O$ draw a perpendicular to the radius vector $OP$ to meet the tangent at $T$ and the normal at $N$. Then $OT$ is called the polar subtangent and $ON$ is called the polar subnormal to the curve $r = f(\theta)$ at $P$. From the figure, $POT$ is a right-angled triangle.

$\therefore$ With the usual notation $\tan\phi = \frac{OT}{OP} = \frac{OT}{r}$.

$\therefore OT = r\tan\phi = r^2\left(\frac{d\theta}{dr}\right)$. $\quad\left(\because \tan\phi = r\left(\frac{d\theta}{dr}\right)\right)$

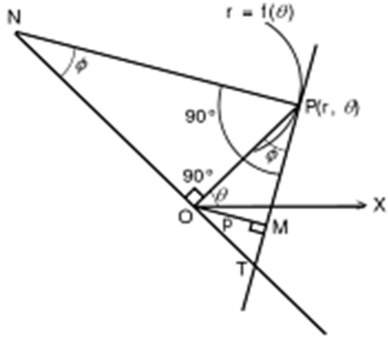
$\therefore$ **Polar sub-tangent** $= r^2\left(\frac{d\theta}{dr}\right)$.

Similarly, we can prove that **polar sub-normal** $ON = \frac{dr}{d\theta}$.

**The polar tangent** $PT = OP\sec\phi$

$$= r\sqrt{1+\tan^2\phi}$$

$\therefore$ Polar tangent $= r\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}$




**Polar normal** $PN = OP \operatorname{cosec} \phi$

$$= r\sqrt{1 + \cot^2 \phi}$$

$$= r\sqrt{1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2}$$

$\therefore$ Polar Normal $= \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$

**Example 1**

Find the equation of the tangent to the parabola $y^2 = 4ax$ at $(x_1, y_1)$.

**Solution**

$y^2 = 4ax$.

Differentiating w. r. t. $x$ we get $2yy' = 4a$.

$\therefore y' = \left(\frac{2a}{y_1}\right)$

$\therefore$ Slope of the tangent at $(x_1, y_1) = \left(\frac{2a}{y_1}\right)$.

Equation of the tangent at $(x_1, y_1)$ is

$y - y_1 = \left(\frac{2a}{y_1}\right)(x - x_1)$.

$\therefore yy_1 - y_1^2 = 2ax - 2ax_1$.

$\therefore yy_1 = 2ax + y_1^2 - 2ax_1$.

$= 2ax + 4ax_1 - 2ax_1 \qquad (\because y_1^2 = 4ax_1)$

$\therefore yy_1 = 2a(x + x_1)$.

**Example 2**

Find the equation of the normal to the curve $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ at $\theta = \pi/2$.

**Solution**

$x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$.

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$

$= \frac{a\sin\theta}{a(1-\cos\theta)}$

$= \cot(\theta/2)$.

$\therefore$ Slope of the tangent at $\theta = \pi/2$ is $\cot(\pi/4) = 1$.

$\therefore$ Slope of the normal at $\theta = \pi/2$ is $-1$.

Also, at $\theta = \pi/2, x = a\left(\frac{\pi}{2} - 1\right)$ and $y = a$.

$\therefore$ Equation of the normal at $\theta = \pi/2$ is

$y - a = (-1)\left[x - a\left(\frac{\pi}{2} - 1\right)\right]$.

(i.e.) $x + y = \frac{a\pi}{2}$.

**Example 3**

Find the lengths of the tangent, normal, subtangent, subnormal to the curve $x = a(\theta - \sin\theta);\ y = a(1 - \cos\theta)$ at $\theta = \pi/2$.

**Solution**

$x = a(\theta - \sin\theta);\ y = a(1 - \cos\theta)$.

$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \cot\left(\frac{\theta}{2}\right)$.

At $\theta = \pi/2, y' = 1$ and $y = a$.

Length of the tangent $= \frac{y\sqrt{1+y'^2}}{y'} = \sqrt{2}a$

Length of the normal $= y\sqrt{1 + y'^2} = \sqrt{2}a$

Length of the subtangent $= \frac{y}{y'} = a$.

Length of the subnormal $= yy' = a$.

**Example 4**

Find the polar subtangent for cardioid $r = a(1 - \cos\theta)$.

**Solution**

$r = a(1 - \cos\theta)$.

$\therefore \frac{dr}{d\theta} = a\sin\theta$.

Polar subtangent $= r^2\frac{d\theta}{dr}$.

$$= \frac{a^2(1-\cos\theta)^2}{a\sin\theta}$$
$$= \frac{4a\sin^4\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$
$$= 2a\sin^2\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right).$$

**Exercise 1**

1. Find the equation of the tangent to the following curves.
   (i) $y^2 = 4ax$ at $(at^2, 2at)$ (ii) $y = 2x^2 - 4x + 5$ at $(3, 11)$
2. Find the equation of the tangent and normal to the following curves at the points indicated.
   (i) $\sqrt{x} + \sqrt{y} = 5$ at $(9, 4)$ (ii) $x = \sin t\,; y = \cos 2t$ at $t = \pi/6$
3. Find the length of the tangent, normal, subtangent, subnormal to the following curves.
   (i) $y^2(2a - x) = x^2$ at $(a, a)$ (ii) $6y^2 = x^3$ at $(6, 6)$.
4. Find the lengths of the polar sub-tangent and the polar sub-normal to the following curves.
   (i) $r\theta = a$ (ii) $r = a(1 + \cos\theta)$ (iii) $\frac{2a}{r} = 1 + e\cos\theta$

**Answers**

1. (i) $yt = x + at^2$ (ii) $8x - y = 13$.
2. (i) $2x + 3y = 30; 3x - 2y = 19$.
   (ii) $4x + 2y = 3; 2x - 4y + 1 = 0$.
3. (i) $\frac{\sqrt{5}a}{2}; \sqrt{5}a; \frac{a}{2}; 2a$ (ii) $2\sqrt{13}; 3\sqrt{13};\ 4; 9$.
4. (i) $a;\ -\frac{a}{\theta^2}$ (ii) $2a\cos^2\left(\frac{\theta}{2}\right)\cot\left(\frac{\theta}{2}\right);\ -a\sin\theta$.
   (iii) $\frac{2a}{e\sin\theta}; \frac{2ae\sin\theta}{(1+e\cos\theta)^2}$

## 2.2 Polar Curves

Let $OX, OY$ be the rectangular axes with origin at $O$. With respect to this system any point P in the plane can be specified by its Cartesian coordinates $(x, y)$. The point P can also be specified by the coordinates $(r, \theta)$ where $r = OP$ and $\theta$ is the angle that the line $OP$ makes with the positive direction of the $x$-axis measured in the anti clockwise direction. The numbers $r$ and $\theta$ are called the polar coordinates of the point $P$. Then $O$ is called the pole, the ray $OX$ is called the radius vector joining $O$ and $P$.

The relations between Cartesian and polar coordinates of a point $P$ are given by

$x = r\cos\theta$ ; $y = r\sin\theta$ and $r = \sqrt{x^2 + y^2};\ \theta = \tan^{-1} x$.

### 2.2.1 Angle between radius and vector and tangent

Let $r = f(\theta)$ be the equation of a curve in polar coordinates.

Let $P(r, \theta)$ be a point on the curve.

Let $\phi$ be the angle between the radius vector $OP$ and the tangent to the curve at P.

Let $\varphi$ be the angle made by the tangent at $P$ with the initial line.

Now with respect to the Cartesian coordinates we have

$x = r\cos\theta = f(\theta)\cos\theta$ (1)

$y = r\sin\theta = f(\theta)\sin\theta$ (2)

Equations (1) and (2) can be taken as the parametric equations of the given curve with $\theta$ as parameter.

We know that the slope of the tangent to the curve at P is

$\tan\varphi = \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{\sin\theta f'(\theta) + f(\theta)\cos\theta}{\cos\theta f'(\theta) - f(\theta)\sin\theta}$

$= \frac{\tan\theta + [f(\theta)/f'(\theta)]}{1 - \tan\theta[f(\theta)/f'(\theta)]}$ (3)

But $\varphi = \theta + \phi$.

Hence $\tan\varphi = \tan(\theta + \phi)$.

$\therefore \tan\varphi = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$

(4)

Comparing (3) and (4) we get

$\tan\varphi = \frac{f(\theta)}{f'(\theta)} = r \div \left(\frac{dr}{d\theta}\right) = r \times \left(\frac{d\theta}{dr}\right)$.

### 2.2.2 The angle of intersection of two polar curves

At a point of intersection of the two given curves the radius vector is the same.

Let $\varphi_1$ and $\varphi_2$ respectively be the angles between the common radius vector and the tangents to the curve.

Then $|\phi_1 - \phi_2|$ is the angle of intersection of the curves.

**Note**

1. If $|\phi_1 - \phi_2| = \frac{\pi}{2}$ then the two curves are said to intersect orthogonally.

2. If $|\phi_1 - \phi_2| = 0$ then the two curves are said to touch each other.

**Example 5**

Find the angle between the radius vector and the tangent to the curve $r = a(1 - \cos\theta)$ at $\theta = \frac{\pi}{6}$. Also find the slope of the tangent at $\theta = \frac{\pi}{6}$.

**Solution**

$r = a(1 - \cos\theta)$.

$\therefore \frac{dr}{d\theta} = a\sin\theta$.

$\therefore \tan\varphi = r \div \frac{dr}{d\theta} = \frac{a(1-\cos\theta)}{a\sin\theta} = \frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$.

$= \tan\left(\frac{\theta}{2}\right)$.

$\therefore \phi = \frac{\theta}{2}$. Hence at $\theta = \frac{\pi}{6}$ we have $\phi = \frac{\pi}{12}$.

We know that $\varphi = \theta + \phi$. Hence $\varphi = \frac{\pi}{6} + \frac{\pi}{12} = \frac{\pi}{4}$.

$\therefore$ Slope $= \tan\varphi = \tan\left(\frac{\pi}{4}\right) = 1$.

**Example 6**

Find the angle of intersection of curves $r = \frac{a}{1+\cos\theta}$ and $r = \frac{b}{1-\cos\theta}$.

**Solution**

Let $P(r, \theta)$ be the point of intersection of the two curves.

Let $\phi_1$ and $\phi_2$ be the angles which $OP$ makes with two tangents at $P$.

We have $r = \frac{a}{(1+\cos\theta)} = \frac{1}{2}a\sec^2\left(\frac{\theta}{2}\right)$.

Taking logarithm and differentiating, we get

$\frac{1}{r}\left(\frac{dr}{d\theta}\right) = \frac{2}{\sec(\theta/2)}\left[\sec\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right)\left(\frac{1}{2}\right)\right]$

$= \tan\left(\frac{\theta}{2}\right)$.

$\therefore \tan\phi_1 = r \div \left(\frac{dr}{d\theta}\right) = \cot\left(\frac{\theta}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$

$\therefore \phi_1 = \frac{\pi}{2} - \frac{\theta}{2}$.

Similarly for the curve $r = \frac{b}{(1-\cos\theta)}$ we can prove $\phi_2 = -\frac{\theta}{2}$.

$\therefore |\phi_1 - \phi_2| = \frac{\pi}{2}$.

Note Hence the two curves intersect orthogonally.

**Example 7**

Find the angle of intersection of the curves $r = a\theta$ and $r\theta = a$.

**Solution**

Solving the two curves we have $\theta^2 = 1$ or $\theta = \pm 1$.

Hence the point of intersection of two curves are $P(a, 1)$ and $Q(-a, -1)$.

For the curve $r = a\theta$ we have $\frac{dr}{d\theta} = a$.

$\therefore \tan\phi = \frac{r}{a} = \frac{a\theta}{a} = \theta$.

$\therefore$ At $P(a, 1)$, $\tan\phi_1 = 1$ and $\phi_1 = \frac{\pi}{4}$.

Similarly for the curve $r\theta = a$ we can prove that $\tan\phi_2 = -1$ at $P(a, 1)$.

Hence $\phi_2 = \frac{3\pi}{4}$.

$\therefore$ The angle between the curves is $|\phi_1 - \phi_2| = \left|\frac{\pi}{4} - \frac{3\pi}{4}\right| = \frac{\pi}{2}$.

Hence at $P(a, 1)$ the curves intersect orthogonally.

Similarly we can prove that the curves intersect orthogonally at $Q(-a, -1)$ also.

**Exercise 2**

1. Find the angle between the radius vector and the tangent for the following curves.

(i) $r = a(1 - \cos\theta)$ (ii) $\frac{2}{r} = 1 + \cos\theta$

(iii) $r^n = b^n(\cos n\theta + \sin n\theta)$

2. Find the angle of intersection of the following curves.

(i) $r = a(1 + \cos\theta); r = b(1 - \cos\theta)$

(ii) $r = a\sin 2\theta\, ; r = a\cos 2\theta$

(iii) $r = a\log\theta\, ; r = \frac{a}{\log\theta}$

**Answers**

1. (i) $\frac{\theta}{2}$ (ii) $\frac{\pi}{2} - \frac{\theta}{2}$ (iii) $\frac{\pi}{4} + n\theta$

2. (i) $\frac{\pi}{2}$ (ii) $\tan^{-1}\left(\frac{4}{3}\right)$ (iii) $\tan^{-1}\left(\frac{2e}{1-e^2}\right)$

**2.3 Pedal Equation of a curve ( p – r equation)**

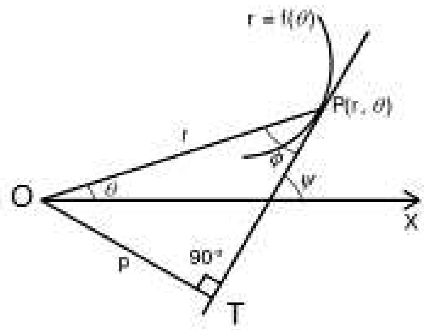
**Length of the perpendicular from the pole to the tangent at $\boldsymbol{P(r,\theta)}$**

Let $p$ denote the length of the perpendicular $OT$ drawn from $O$ to the tangent at $P(r,\theta)$. Let $\phi$ be the angle between the tangent at $P$ and the radius vector $OP$.

From triangle $OTP$, $p = r\sin\phi$.

$\therefore \frac{1}{p^2} = \frac{\operatorname{cosec}^2\phi}{r^2} = \frac{1+\cot^2\phi}{r^2}.$

$= \frac{1}{r^2} + \frac{1}{r^2}\left(\frac{1}{\tan\phi}\right)^2.$




$= \frac{1}{r^2} + \frac{1}{r^2}\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2$

$\therefore \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$

**Note** If we put $r = \frac{1}{u}$ then $\left(\frac{dr}{d\theta}\right) = -\frac{1}{u^2}\left(\frac{du}{d\theta}\right)$ and the above equation becomes $\frac{1}{p^2} = u^2 + u^4\left[-\frac{1}{u^2}\left(\frac{du}{d\theta}\right)\right]^2.$

$\therefore \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2.$

**Definition**

The equation of a curve in terms of $p$ and $r$ is called the p – r equation of the curve. (pedal equation)

Let the equation of the given curve be $r = f(\theta)$ (1)

For this curve we have $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$ (2)

Eliminating $\theta$ between (1) and (2) we get the pedal equation of the given curve.

**Example 8**

Find the $p - r$ equation of $r = a\sin\theta$.

Solution

$r = a\sin\theta$.

$\therefore \frac{dr}{d\theta} = a\cos\theta$.

We have $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

$= \frac{1}{r^2} + \frac{1}{r^4}(a\cos\theta)^2$

$= \frac{1}{r^2} + \frac{a^2\cos^2\theta}{r^4}$

$= \frac{a^2\sin^2\theta + a^2\cos^2\theta}{r^4}$

$= \frac{a^2}{r^4}$.

$\therefore a^2p^2 = r^4$.

$\therefore ap = r^2$ is the required $p - r$ equation.

**Example 9**

Find the $p - r$ equation of the curve $r^2\sin 2\theta + a^2 = 0$.

**Solution**

$r^2\sin 2\theta + a^2 = 0$.

$\therefore r^2 = -a^2\operatorname{cosec} 2\theta$

(1)

Differentiating (1) with respect to $\theta$ we get

$2r\frac{dr}{d\theta} = 2a^2\operatorname{cosec} 2\theta\cot 2\theta$

$\therefore \frac{dr}{d\theta} = \frac{a^2\operatorname{cosec} 2\theta\cot 2\theta}{r}$.

We have $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

$= \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{a^4\operatorname{cosec}^2 2\theta\cot^2 2\theta}{r^2}\right)$

$= \frac{1}{r^2} + \frac{1}{r^6}(a^4\operatorname{cosec}^2 2\theta)(\operatorname{cosec}^2 2\theta - 1)$

$= \frac{1}{r^2} + \frac{1}{r^6}\left[r^4\left(\frac{r^4}{a^4} - 1\right)\right]$ (using (1))

$= \frac{1}{r^2} + \frac{1}{r^2}\left(\frac{r^4 - a^4}{a^4}\right)$

$$= \frac{1}{r^2} + \frac{r^2}{a^4} - \frac{1}{r^2}$$

$\therefore \frac{1}{p^2} = \frac{r^2}{a^4}$.

$\therefore p^2 r^2 = a^4$ or $pr = a^2$.

**Example 10**

Find the $p - r$ equation of the curve $r = \frac{a}{2}(1 - \cos\theta)$

**Solution**

$$r = \frac{a}{2}(1 - \cos\theta) \qquad (1)$$

Differentiating (1) with respect to $\theta$ we get $\frac{dr}{d\theta} = \frac{a}{2}\sin\theta$

We have $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

$$= \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{a^2}{4}\sin^2\theta\right)$$

$$= \frac{4r^2 + a^2\sin^2\theta}{4r^4}$$

$$= \frac{4\left(\frac{a^2}{4}\right)(1-\cos\theta)^2 + a^2\sin^2\theta}{4r^4}$$

$$= \frac{4a^2\sin^4\left(\frac{\theta}{2}\right) + 4a^2\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)}{4r^4}$$

$$= \frac{4a^2\sin^2\left(\frac{\theta}{2}\right)\left[\sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)\right]}{4r^4}$$

$$= \frac{a^2\sin^2\left(\frac{\theta}{2}\right)}{r^3\left(\frac{a}{2}\right)(1-\cos\theta)} \qquad \text{(From (1))}$$

$$= \frac{a^2\sin^2\left(\frac{\theta}{2}\right)}{r^3\left(\frac{a}{2}\right)\times 2\sin^2\left(\frac{\theta}{2}\right)} = \frac{a}{r^3}.$$

$\therefore p^2 = \frac{a}{r^3}$.

**Example 11**

Find the $p - r$ equation of the conic $\frac{l}{r} = 1 + e\cos\theta$

**Solution**

$$\frac{l}{r} = 1 + e\cos\theta \qquad (1)$$

$\therefore r = \frac{l}{1 + e\cos\theta}$

$\therefore \frac{dr}{d\theta} = \frac{le\sin\theta}{(1+e\cos\theta)^2}$

We have $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left[\frac{l^2 e^2 \sin^2\theta}{(1+e\cos\theta)^4}\right]$

$= \frac{1}{r^2} + \frac{1}{r^4}\left[\frac{l^2 e^2 \sin^2\theta}{(l/r)^4}\right]$ (Using (1))

$\therefore \frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2 \sin^2\theta}{l^2}$

(2)

Now, $1 + e\cos\theta = \frac{l}{r} \Rightarrow e\cos\theta = \frac{l}{r} - 1$

$\Rightarrow e^2\cos^2\theta = \left(\frac{l}{r} - 1\right)^2$

$\Rightarrow e^2(1 - \sin^2\theta) = \left(\frac{l}{r} - 1\right)^2$

$\Rightarrow e^2\sin^2\theta = e^2 - \left(\frac{l}{r} - 1\right)^2$

From (2) we get $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{l^2}\left[e^2 - \left(\frac{l}{r} - 1\right)^2\right]$

$= \frac{1}{r^2} + \frac{1}{l^2}\left[\frac{e^2r^2 - l^2 - r^2 + 2lr}{r^2}\right]$

$= \frac{e^2r^2 - r^2 + 2lr}{l^2r^2}$

$= \frac{e^2r - r + 2l}{l^2r}$

$\therefore p^2 = \frac{l^2r}{e^2r - r^2 + 2l}$

This is the $p - r$ equation of the conic $\frac{l}{r} = 1 + e\cos\theta$.

**Example 12**

Find the $p - r$ equation of the curve $r = a\, e^{\theta\cot\alpha}$.

**Solution**

$\frac{dr}{d\theta} = (\cot\alpha)a\, e^{\theta\cot\alpha}$.

We have $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

$= \frac{1}{r^2} + \frac{1}{r^4}\left(a^2 e^{2\theta\cot\alpha}\right)(\cot^2\alpha)y^2.$

$= \frac{r^2 + a^2\cot^2\alpha\, e^{2\theta\cot\alpha}}{r^4}$

$= \frac{r^2 + \cot^2\alpha\left(a\, e^{\theta\cot\alpha}\right)^2}{r^4}$

$= \frac{r^2 + r^2\cot^2\alpha}{r^4}$

$$= \frac{\operatorname{cosec}^2 \alpha}{r^2}$$

$\therefore p^2 = r^2 \sin^2 \alpha.$

Hence $p = r \sin \alpha.$

**Exercise 3**

Find the $p - r$ equation of the following curves.

(i) $2a = r(1 - \cos\theta)$

(ii) $r^2 = a^2 \sin 2\theta$

(iii) $r^2 \cos 2\theta = a^2$

**Answers**

(i) $p^2 = ar$ (ii) $pa^2 = r^2$ (iii) $pr = a^2$

## 2.4 Curvature

### 2.4.1 The length of an arc and its derivatives

Consider a curve given by the equation $y = f(x)$. Let $A$ be a fixed point on the curve. Let $P(x, y)$ be an arbitrary point on the curve. Let $s$ denote the arc length of $P$. Clearly $s$ is a function of $x$.

We shall now prove that $\frac{ds}{dx} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}.$

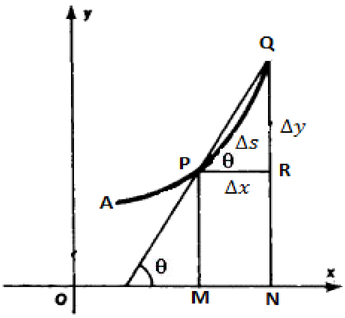
Let $Q(x + \Delta x, y + \Delta y)$ be a neighbouring point on the curve. Let arc $AQ = s + \Delta s$.

From the right angled triangle $PQR$ we get $PQ^2 = PR^2 + RQ^2$.

$$= (\Delta x)^2 + (\Delta y)^2$$

$\therefore \left(\frac{PQ}{\Delta s}\right)^2 \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2.$

Now as $P \to Q, \frac{PQ}{\Delta s} = \left(\frac{chord\ PQ}{arc\ PQ}\right) \to 1$ and $\Delta x \to 0$.




$\therefore \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2.$

$\therefore \frac{ds}{dx} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}.$

**Note** Let the tangent at $P$ and the chord $PQ$ make angle $\phi$ and $\theta$ respectively with the $x$-axis. From the right angled triangle $PQR$ we get

$\cos\theta = \frac{PR}{PQ} = \frac{\Delta x}{PQ} = \left(\frac{\Delta x}{\Delta s}\right)\left(\frac{\Delta s}{PQ}\right)$

Now, as $P \to Q, \theta \to \phi$ and $\frac{\Delta s}{PQ} = \left(\frac{arc\ PQ}{ChordP\ PQ}\right) \to 1.$

$\therefore \cos\phi = \frac{dx}{ds}$. Similarly $\sin\phi = \frac{dy}{ds}$

Also $\sin\phi = \frac{dy}{ds} = \left(\frac{dy}{dx}\right)\left(\frac{dx}{ds}\right) = \frac{y_1}{(1+y_1^2)^{\frac{1}{2}}}$ and $\sin\phi = \frac{dx}{ds} = \frac{1}{(1+y_1^2)^{\frac{1}{2}}}$

**Definition**

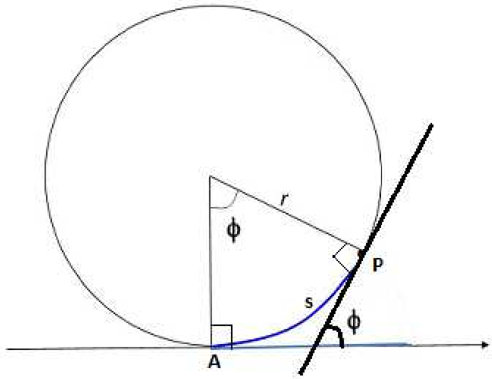
Consider a curve given by the equation $y = f(x)$. suppose the curve has a definite tangent at each point. Let $A$ be a fixed point on the curve and $P$ be an arbitrary point on the curve. Let $s$ denote the arc length $AP$. Let $\phi$ be the angle made by the tangent with the $x$-axis. Then $\left(\frac{d\phi}{ds}\right)$ is called the curvature of the curve at $P$.

**Thus the curvature is the rate of turning of the tangent w. r. t. the arc length**.

**Theorem 2.1**

The curvature of a circle of radius $r$ at any point is $\frac{1}{r}$.

**Proof**




Let $A$ be a fixed point on the circle and $P$ be any point on the circle. Let arc $AP = s$. Let the tangent at $P$ make an angle $\phi$ with the tangent at $A$. Then $\angle AOP = \phi$.

$\therefore s = r\phi$.

$\therefore \frac{ds}{d\phi} = r$ and hence $\frac{d\phi}{ds} = \frac{1}{r}$.

Thus the curvature of a circle of radius $r$ is $\frac{1}{r}$.

**Definition**

The reciprocal of the curvature of a curve at any point is called the radius of curvature at that point and it is denoted by $\rho$.

Hence we have $\rho = \frac{ds}{d\phi}$.

**Note** For a circle of radius $r$, the radius of curvature at any point is equal to $r$.

### 2.4.2 Formula for radius of curvature

### 1. Cartesian Form

We know that $\frac{dy}{dx} = \tan\phi$.

Differentiating w. r. t. $s$ we get,

$\left(\frac{d^2y}{dx^2}\right)\left(\frac{dx}{ds}\right) = \sec^2\phi\left(\frac{d\phi}{ds}\right)$.

$\therefore y_2 \cos\phi = \sec^2\phi\left(\frac{d\phi}{ds}\right)$.

$\therefore \frac{ds}{d\phi} = \frac{\sec^3 \phi}{y_2} = \frac{(1+\tan^2 \phi)^{\frac{3}{2}}}{y_2}.$

$\therefore \ \rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}.$

**Note** If we start with the equation $\cot \phi = \frac{dx}{dy}$ and proceed as above we arrive at the following alternative formula for the radius of curvature.

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\left(\frac{d^2x}{dy^2}\right)}.$$

This alternative formula can be used to find $\rho$ when $\frac{dy}{dx}$ becomes $\infty$ at the given point.

### 2. Parametric Form

Let $x = f(t)$ and $y = g(t)$ be the parametric equations of the given curve.

$\therefore \frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) = \frac{g'(t)}{f'(t)}$ (1)

Where $f'(t) = \frac{df}{dt}$ and $g'(t) = \frac{dg}{dt}$

$\frac{d^2y}{dx^2} = \frac{g''f'-g'f''}{(f')^2}$ (2)

Substituting (1) and (2) in the Cartesian form and simplifying we get

$$\rho = \frac{\left(f'^2+g'^2\right)^{\frac{3}{2}}}{f'g''-f''g'}$$

### 3. Implicit Form

Let $f(x,y) = 0$ be the implicit form of the given curve. Differentiating $f(x,y) = 0$ we get $f_x + f_y y' = 0$.

$\therefore y' = -\frac{f_x}{f_y}.$

(1)

Now, $y'' = -\frac{1}{(f_y)^2}\left[f_y\left(f_{xx} + f_{yx}y'\right) - f_x\left(f_{xy} + f_{yy}y'\right)\right].$

By (1), $y'' = -\frac{1}{(f_y)^2}\left[f_{xx}\left(f_y\right)^2 - 2f_{xy}f_xf_y + f_{yy}(f_x)^2\right]$ (2)

Substituting (1) and (2) in the Cartesian form and simplifying we get,

$$\rho = \frac{(f_x^2+f_y^2)^{\frac{3}{2}}}{f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2}.$$

**4. Polar Form**

Let $r = f(\theta)$ be the given curve in polar coordinates.

$\therefore x = r\cos\theta$ and $y = r\sin\theta$, may be regarded as the parametric equations of the given curve the parameter being $\theta$.

$\therefore \frac{dx}{d\theta} = \cos\theta\frac{dr}{d\theta} - r\sin\theta$ and $\frac{dy}{d\theta} = \sin\theta\frac{dr}{d\theta} + r\sin\theta$

$\therefore \frac{d^2x}{d\theta^2} = \cos\theta\frac{d^2r}{d\theta^2} - 2\sin\theta\frac{dr}{d\theta} - r\cos\theta$ and

$\frac{d^2y}{d\theta^2} = \sin\theta\frac{d^2r}{d\theta^2} + 2\cos\theta\frac{dr}{d\theta} - r\sin\theta$

Substituting these values in the formula for $\rho$ in parametric form and simplifying we get

$\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-r\,r_2}$ where $r_1 = \frac{dr}{d\theta}$ and $r_2 = \frac{d^2r}{d\theta^2}$.

**5. $p - r$ Form**

With the usual notations we prove the following.

(i) $\left(\frac{ds}{dr}\right)^2 = r^2\left(\frac{d\theta}{dr}\right)^2 + 1.$

(ii) $\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$

(iii) $\sin\varphi = r\frac{d\theta}{ds}$

(iv) $\cos\varphi = \frac{dr}{ds}$

**Proof**

Let $y = f(x)$ and $s$ be the arc length.

Then $\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$

(i.e.) $dx^2 + dy^2 = ds^2$ (1)

Now, $x = r\cos\theta$ and $y = r\sin\theta$.

Taking the total differentials we get,

$dx = -r\sin\theta\ d\theta + \cos\theta\ dr$

$dy = r\cos\theta\ d\theta + \sin\theta\ dr.$

$\therefore dx^2 + dy^2 = r^2d\theta^2 + dr^2$

$\therefore ds^2 = r^2d\theta^2 + dr^2$ (using (1)) (2)

$\therefore \left(\frac{ds}{dr}\right)^2 = r^2\left(\frac{d\theta}{dr}\right)^2 + 1$. Hence we get (i).

Also from (2), $\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$. Hence we get (ii).

Let $\varphi$ be the angle between the tangent at $P(r,\theta)$ and the radius vector $r$. Let $\psi$ be the angle which the tangent makes with the initial line. Let $p$ be the length of the perpendicular drawn from the origin to the tangent at $P$.

Then we have $\tan\varphi = r \div \frac{dr}{d\theta}$.

$\therefore \cot\varphi = \frac{1}{r}\left(\frac{dr}{d\theta}\right)$

From (ii) we get $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$.

$$= r\sqrt{1 + \frac{1}{r}\left(\frac{dr}{d\theta}\right)^2}\ .$$

$$= r\sqrt{1 + \cot^2\varphi}$$

$$= r\operatorname{cosec}\varphi.$$

$\therefore \sin\varphi = r\frac{d\theta}{ds}$. Hence we get (iii).

From (i) we get, $\frac{ds}{dr} = \sqrt{r^2\left(\frac{d\theta}{dr}\right)^2 + 1}$

$$= \sqrt{\tan^2\varphi + 1} = \sec\varphi.$$

$\therefore \cos\varphi = \frac{dr}{ds}$.

### 2.4.3 Formula for radius of curvature in $p - r$ coordinates

We have (i) $\sin\varphi = r\frac{d\theta}{ds}$

(ii) $\cos\varphi = \frac{dr}{ds}$

(iii) $p = r\sin\varphi$.

Now from (iii) we get $\frac{dp}{dr} = \sin\varphi + r\cos\varphi\frac{d\varphi}{dr}$.

$$= r\frac{d\theta}{ds} + r\frac{dr}{ds}\frac{d\varphi}{dr} = r\frac{d\theta}{ds} + r\frac{d\varphi}{ds}.$$

$$= r\left(\frac{d\theta}{ds} + \frac{d\varphi}{ds}\right) = r\frac{d}{ds}(\theta + \varphi).$$

$\frac{dp}{dr} = r\frac{d\psi}{ds}$.

$\therefore \frac{ds}{d\psi} = r\frac{dr}{dp}$.

$\therefore\ \rho = r\frac{dr}{dp}$.

**Example 13**

Find the radius of curvature at $x = \frac{\pi}{2}$ on the curve $y = \sin x$.

**Solution**

$y = \sin x.$

$\therefore y_1 = \cos x$ and $y_2 = -\sin x.$

$\therefore$ At $x = \frac{\pi}{2}, y_1 = 0$ and $y_2 = -1.$

$\therefore \ \rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1+0)^{\frac{3}{2}}}{-1} = -1 \ .$

**Example 14**

Find the radius of curvature at any point of the curve $x = a\cos^3\theta\,, y = a\sin^3\theta.$

**Solution**

We have $x = a\cos^3\theta\,, y = a\sin^3\theta.$

$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}.$

$= \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}.$

$\therefore \frac{d^2y}{dx^2} = \frac{d}{d\theta}(-\tan\theta)\frac{d\theta}{dx}.$

$= \frac{-\sec^2\theta}{-3a\cos^2\theta\sin\theta}.$

$= \frac{1}{3a\cos^4\theta\sin\theta}.$

$\therefore \ \rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = (1+\tan^2\theta)^{\frac{3}{2}}\,(3a\cos^4\theta\sin\theta = 3a\cos\theta\sin\theta.$

**Example 15**

Find the radius of curvature of the curve given by $x^3 - 2x^2y + 3xy^2 - 4y^3 + 5x^2 - 6xy + 7y^2 - 8y = 0.$

**Solution**

Let $f(x,y) = x^3 - 2x^2y + 3xy^2 - 4y^3 + 5x^2 - 6xy + 7y^2 - 8y = 0.$

$\therefore f_x = 3x^2 - 4xy + 3y^2 + 10x - 6y.$

$f_y = -2x^2 + 6xy - 12y^2 - 6x + 14y - 8.$

$f_{xx} = 6x - 4y + 10.$

$f_{yy} = 6x - 24y + 14.$

$f_{xy} = -4x + 6y - 6.$

At $(0,0), f_x = 0; f_y = -8; f_{xx} = 10; f_{yy} = 14$ and $f_{xy} = -6.$

Substituting these values in the formula for $\rho$ in implicit form we get

$$\therefore \rho = \frac{(f_x^2+f_y^2)^{\frac{3}{2}}}{f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2}$$

$$\rho = \frac{(0^2+(-8)^2)^{\frac{3}{2}}}{10(-8)^2-2(-6)(0)(-8)+14(0)^2}$$

$$= \frac{(8^2)^{\frac{3}{2}}}{10\times 64} = \frac{4}{5}.$$

**Example 16**

Find the pedal equation of the curve $x^2 + y^2 = 2\,ax$ and deduce its radius of curvature.

**Solution**

Obviously the given equation represents the equation of a circle. Put $x = r\cos\theta\,, y = r\sin\theta$.

$\therefore r^2 = 2ar\cos\theta$.

$\therefore r = 2a\cos\theta$ which is the polar equation of the circle.

Now, $\frac{dr}{d\theta} = -2a\sin\theta$.

We have $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$.

$$= \frac{1}{r^2} + \frac{4a^2\sin^2\theta}{r^4}.$$

$$= \frac{r^2+4a^2\sin^2\theta}{r^4}.$$

$$= \frac{4a^2\cos^2\theta+4a^2\sin^2\theta}{r^4}.$$

$$= \frac{4a^2}{r^4}.$$

$\therefore p^2 = \frac{r^4}{4a^2}$.

Differentiating (1) w. r. t. $r$ we get $2p\frac{dp}{dr} = \frac{4r^3}{4a^2}$.

$\therefore \frac{dp}{dr} = \frac{r^3}{2a^2p}$.

$\therefore r\frac{dr}{dp} = r\left(\frac{2a^2p}{r^3}\right)$.

$$= \left(\frac{2a^2}{r^2}\right)\left(\frac{r^2}{2a}\right) \qquad \text{( using (1))}$$

$\therefore \ \rho = a$.

**Example 17**

Find the radius of curvature of the curve $r = a\theta$.

Solution

For the polar form the radius of curvature is

$\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-r\,r_2}$ where $r_1 = \frac{dr}{d\theta}$ and $r_2 = \frac{d^2r}{d\theta^2}$.

For the curve $r = a\theta$ we have $r_1 = a$ and $r_2 = 0$.

$\therefore \rho = \frac{(a^2\theta^2+a^2)^{\frac{3}{2}}}{a^2\theta^2+2a^2} = \frac{a^3(\theta^2+1)^{\frac{3}{2}}}{a^2(\theta^2+2)}$

$\therefore \rho = \frac{a(\theta^2+1)^{\frac{3}{2}}}{(\theta^2+2)}$.

**Example 18**

Find the radius of curvature for the general conic. $\frac{l}{r} = 1 + e\cos\theta$

**Solution**

We know that $p - r$ equation of the curve $\frac{l}{r} = 1 + e\cos\theta$ is

$p^2 = \frac{l^2 r}{e^2r-r+2l}$ (1)

The radius of curvature is $\rho = r\frac{dr}{dp}$.

Differentiating (1) with respect to $r$ we get

$2p\frac{dp}{dr} = l^2\left[\frac{(e^2r-r+2l)-r(e^2-1)}{(e^2r-r+2l)^2}\right]$.

$= \frac{2l^3}{(e^2r-r+2l)^2}$.

$\therefore \frac{dr}{dp} = \frac{p\left(e^2r-r+2l\right)^2}{l^3}$.

$\therefore r\frac{dr}{dp} = \frac{rp\left(e^2r-r+2l\right)^2}{l^3}$.

$\therefore \rho = \left[\frac{r\left(e^2r-r+2l\right)^2}{l^3}\right] \times p$.

$= \left[\frac{r\left(e^2r-r+2l\right)^2}{l^3}\right] \times \left[\frac{l^2r}{e^2r-r+2l}\right]^{\frac{1}{2}}$ (using (1))

$= \frac{r^{\frac{3}{2}}(e^2r-r+2l)^{\frac{3}{2}}}{l^2}$.

$\therefore \rho = \frac{(e^2r^2-r^2+2lr)^{\frac{3}{2}}}{l^2}$.

**Exercise 4**

1. Find the curvature of the following curves at the indicated points.

(i) $2y = x - x^2 + x^3$ at $\left(1, \frac{1}{2}\right)$  (ii) $xy = 12$ at $(3, 4)$

2. Find the radius of curvature of the following curves at the indicated points.

(i) $y = \frac{\log x}{x}$ at $x = 1$.  (ii) $\sqrt{x} + \sqrt{y} = 1$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$

(iii) $x^2 = 4ay$ at $(0, 0)$  (iv) $y = e^x$ at $(0, 1)$

3. Find the radius of curvature of the following

(i) $x = a\log(\sec\theta + \tan\theta)\,; y = \sec\theta$

(ii) $x = a(\cos t + t\sin t);\ y = a(\sin t - t\cos t)$

(iii) $x = 3t^3; y = 3t - t^3$ at $t = 1$.

4. Find the pedal equation of the curve $r^n = a^n\cos n\theta$ and hence find $\rho$.

**Answers**

1. (i) $\frac{1}{\sqrt{2}}$  (ii) $\frac{24}{125}$  2. (i) $\frac{2\sqrt{2}}{3}$  (ii) $\frac{1}{\sqrt{2}}$  (iii) $2a$  (iv) $2\sqrt{2}$

3. (i) $a\sec^2\theta$  (ii) $at$  (iii) 6  4. $\frac{a^n r^{1-n}}{n+1}$

**2.5 Evolutes**

**Centre and circle of curvature**

**Definition**

Consider a point $P$ on any given curve. Draw the normal to the curve at $P$. Let $C$ be the point on the normal to the curve at $P$ such that $CP = \rho$ and $C$ lies on the side towards which the curve is concave. Then $C$ is called the centre of curvature to the curve at $P$. The circle with centre $C$ and radius $\rho$ is called the circle of the circle of curvature at $P$.

**Coordinates of the centre of curvature**

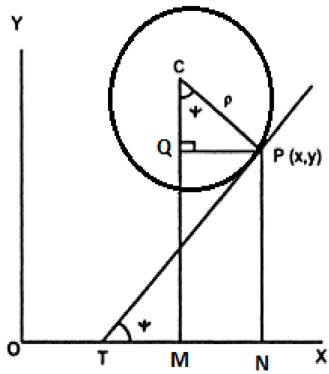
Let $y = f(x)$ be the given curve. Let $P(x, y)$ be any point on the curve. Let $C(\alpha, \beta)$ be the centre and $\rho$ the radius of curvature of the curve at $P$. Let $\psi$ be the angle made by the tangent to the curve at $P$ with the positive direction of the $x$-axis. From the figure

$$\alpha = OM = ON - MN = ON - OP$$

$$= x - \rho\sin\psi$$

$$= x - \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}\left(\frac{y_1}{(1+y_1^2)^{\frac{1}{2}}}\right)$$

$$= x - \frac{y_1}{y_2}(1 + y_1^2).$$




$$\beta = MC = MQ + QC$$
$$= y + \rho \cos \psi$$
$$= y + \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}\left(\frac{1}{(1+y_1^2)^{\frac{1}{2}}}\right)$$
$$= y + \frac{1+y_1^2}{y_2}$$

$\therefore$ C is $\left(x - \frac{y_1}{y_2}(1 + y_1^2), y + \frac{1+y_1^2}{y_2}\right)$.

**Definition**

The locus of the centre's of curvature of a curve is called the evolute of the curve.

**Example 19**

Prove that the $y$-coordinate of the centre of the curvature of the curve at the point $(c, c)$ is $2c$.

**Solution**

$xy = c^2$ (1)

Differentiating (1) with respect to $x$ we get

$xy_1 + y = 0$

(2)

$\therefore y_1 = -\frac{y}{x}$

$\therefore$ At $(c, c), y_1 = -1$.

Differentiating (2) with respect to $x$ again we get

$xy_2 + 2y_1 = 0$. Hence $y_2 = -\frac{2y_1}{x}$.

$\therefore$ At $(c, c), y_2 = -\frac{2(-1)}{c} = \frac{2}{c}$.

The $y$-coordinate of the centre of curvature at $(x, y)$ is

$\beta = y + \frac{1+y_1^2}{y_2}$.

$\therefore$ At $(c, c), \beta = c + \frac{1+(-1)^2}{(2/c)} = c + c = 2c$.

**Example 20**

Find the $x$ coordinate of the centre of curvature of the curve $x = at^2, y = 2at$.

**Solution**

$y_1 = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2a}{2at} = \frac{1}{t}$.

$y_2 = -\frac{1}{t^2} \times \frac{1}{2at} = -\frac{1}{2at^3}$.

The $x$-coordinate $\alpha$ of the centre of curvature of the curve is

$\alpha = x - \frac{y_1}{y_2}(1 + y_1^2)$.

$\therefore\ \alpha = at^2 - \frac{1}{t} \times (-2at^3)\left(1 + \frac{1}{t^2}\right)$

$= at^2 + 2at^2\left(1 + \frac{1}{t^2}\right) = 3at^2 + 2a$.

**Example 21**

Find the centre of curvature of $y = x^2$ at the origin.

**Solution**

We have $y = x^2$

$\therefore y_1 = 2x$ and $y_2 = 2$.

$\therefore$ At $(0, 0), y_1 = 0$ and $y_2 = 2$.

Let $(\alpha, \beta)$ be the centre of the curvature at $(0, 0)$.

$\therefore\ \alpha = x - \frac{y_1}{y_2}(1 + y_1^2) = 0$.

$\beta = y + \frac{1+y_1^2}{y_2} = \frac{1}{2}$.

$\therefore$ Centre of curvature is $\left(0, \frac{1}{2}\right)$.

**Example 22**

Find the evolute of the curve given by $x = a\cos^3\theta$ and $y = a\sin^3\theta$.

**Solution**

We have $x = a\cos^3\theta$ and $y = a\sin^3\theta$.

$\therefore y_1 = -\tan^3\theta$ and $y_2 = \frac{1}{3a}\sec^4\theta\,\mathrm{cosec}\,\theta$

Let $(\alpha, \beta)$ be the centre of curvature.

$\therefore\ \alpha = x - \frac{y_1}{y_2}(1 + y_1^2).$

$= a\cos^3\theta + \frac{3a\tan\theta(1+\tan^2\theta)}{\sec^4\theta\,\mathrm{cosec}\,\theta}$

$= a\cos^3\theta + 3a\sin^2\theta\cos\theta.$ (1)

$\beta = y + \frac{1+y_1^2}{y_2}.$

$= a\sin^3\theta + \frac{3a(1+\tan^2\theta)}{\sec^4\theta\,\mathrm{cosec}\,\theta}$

$= a\sin^3\theta + 3a\cos^2\theta\sin\theta$ (2)

Now, to find the equation of the evolute, we have to eliminate $\theta$ from (1) and (2). From (1) and (2), we have

$\alpha + \beta = a(\cos\theta + \sin\theta)^3.$

$\alpha - \beta = a(\cos\theta - \sin\theta)^3.$

$(\alpha+\beta)^{\frac{2}{3}} + (\alpha-\beta)^{\frac{2}{3}} = a^{\frac{2}{3}}(2) = 2a^{\frac{2}{3}}.$

The locus of $(\alpha, \beta)$ is $(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$

**Example 23**

Find the evolute of the parabola $y^2 = 4ax$.

**Solution**

We have $y^2 = 4ax$.

(1)

$\therefore y_1 = \frac{2a}{y}$ and $y_2 = -\frac{4a^2}{y^3}$.

Let $(\alpha, \beta)$ be the centre of curvature.

$\therefore\ \alpha = x - \frac{y_1}{y_2}(1 + y_1^2).$

$= x + \frac{y^2+4a^2}{2a}$

$= 3x + 2a$ (by (1)) (2)

$\beta = y + \frac{1+y_1^2}{y_2}.$

$= -\frac{y^3}{4a^2}.$

$= -\frac{2x^{\frac{2}{3}}}{\sqrt{a}}$ (by (1)) (3)

From (2) and (3) eliminating $x$, we have

$\beta^2 = \frac{4x^3}{a} = \frac{4(\alpha-2a)^3}{27a}$.

$\therefore 27a\,\beta^2 = 4(\alpha - 2a)^3$.

$\therefore$ The locus of $(\alpha, \beta)$ is $27a\,y^2 = 4(x - 2a)^3$.

**Example 24**

The normal to a given curve is tangent to its evolute.

**Solution**

We know that the coordinates of the centre of curvature of the given curve are given by $\alpha = x - \frac{y_1}{y_2}(1 + y_1^2); \beta = y + \frac{1+y_1^2}{y_2}$.

$\therefore \frac{d\alpha}{dx} = 1 - \left(\frac{y_1}{y_2}\right)\ 2y_1y_2 - (1 + y_1^2)\left[\frac{y_2^2 - y_1y_3}{y_2^2}\right]$.

$= 1 - 2y_1^2 - (1 + y_1^2)\left(1 - \frac{y_1y_3}{y_2^2}\right)$.

$= -3y_1^2 + \frac{y_1y_3}{y_2^2} + \frac{y_1^3y_3}{y_2^2} = -\frac{y_1}{y_2^2}(3y_1y_2^2 - y_3 - y_1^2y_3)$.

Now, $\frac{d\beta}{dx} = y_1 + \left[\frac{2y_1y_2^2 - (1+y_1^2)y_3}{y_2^2}\right] = \frac{1}{y_2^2}(3y_1y_2^2 - y_3 - y_1^2y_3)$.

$\therefore \frac{d\beta}{d\alpha} = -\frac{1}{y_1}$.

(1)

But $\frac{d\beta}{d\alpha}$ is the slope of the tangent to the evolute and $y_1$ is the slope of the tangent to the given curve at the corresponding point and their product is $-1$ by (1).

$\therefore$ Tangent to the evolute is normal to the given curve.

**Exercise 5**

1. Find the coordinates of the centre of curvature at the indicated points.

(i) $y = x^2$ at $\left(\frac{1}{2}, \frac{1}{4}\right)$ (ii) $xy = c^2$ at $(c, c)$

(iii) $x = a(\cos t + t\sin t); y = a(\sin t - t\cos t)$ at '$t$'

(iv) $y = x\log x$ at the point where $y' = 0$.

2. Show that the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$.

3. Find the evolutes of the following curves.

(i) $2xy = a^2$

(ii) $x = a[\cos t + \log\tan\left(\frac{t}{2}\right)];\ y = a\sin t$.

(iii) $x = a(\theta - \sin\theta);\ y = a(1 - \cos\theta)$.

**Answers**

1. (i) $\left(-\frac{1}{2}, \frac{3}{4}\right)$ (ii) $(2c, 2c)$ (iii) $(a\cos t, a\sin t)$ (iv) $\left(\frac{1}{e}, 0\right)$

3. (i) $(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$ (ii) $y = a\cosh\left(\frac{x}{a}\right)$

(iii) $x = a(\theta - \sin\theta);\ y = a(1 - \cos\theta)$.

## 2.6 Envelopes

### Introduction

Some curves can be obtained as the envelope of a family of curves. In fact any curve is envelope of the family of all the tangents to the curves.

In this section we discuss the method of finding the envelope of a given family of curves.

### Envelopes – One parameter family of curves

### Definition

Consider an equation of the form $f(x, y, \alpha) = 0$ (1)

For any particular value of $\alpha$ equation (1) represents a particular curve. For different values of $\alpha$ we get a family of curves and the equation (1) is said to represent a one parameter family of curves with $\alpha$ as parameter.

### Examples

1. The equation $y^2 = 4ax$ represents a family of parabolas with a common axis and vertex. Here $a$ is the parameter.

2. The equation $x^2 + y^2 - 2ax = 0$ represents a family of circles with their centres lying on the $x$-axis and passing through the origin. Here $a$ is the parameter.

3. The equation $y = mx + \frac{a}{m}$ represents a family of straight lines where $m$ is the parameter and $a$ is a given constant.

**Note** Let $f(x, y, \alpha) = 0$ represent a one parameter family of curves. Then the curves corresponding to two adjacent values of $\alpha$ need not intersect. For examples $x^2 + y^2 = a^2$ represents a family of concentric circles with centre at origin and any two curves of the family do not intersect.

**Definition**

Let $f(x, y, \alpha) = 0$ be a one parameter family of curves such that any two curves correspond to adjacent values of $\alpha$ intersect.

Now, consider two curves of the family given by $f(x, y, \alpha) = 0$ and $f(x, y, \alpha + h) = 0$ where $h$ is small. As $h \to 0$ the points of intersection of the above two curves will tend to a limiting position and the locus of all these limiting positions is called the envelope of the given family of curves.

**Analytical method of finding envelopes**

Let the equation of a one parameter family of curves be

$f(x, y, \alpha) = 0$

(1)

Consider two adjacent members of the family given by $f(x, y, \alpha) = 0$ and $f(x, y, \alpha + h) = 0$. The points of intersection of these curves satisfy both these equations and hence satisfy the equation

$\frac{f(x,y,\alpha+h)-f(x,y,\alpha)}{h} = 0.$

∴ The coordinates of the limiting positions of the point of intersection satisfy the equation

$\lim_{h\to 0} \frac{f(x,y,\alpha+h)-f(x,y,\alpha)}{h} = 0$ (i.e.) $\frac{\partial f}{\partial x} = 0.$ (2)

∴ The envelope is founded by eliminating $\alpha$ between (1) and (2).

If the given family of curves involves two parameters and further parameters are connected by a relation then the analytic method of finding the envelopes is given below,

Consider a two parameter family of curves given by $f(x, y, a, b) = 0$ (1)

Suppose the parameters $a$ and $b$ are connected by the relation $g(a, b) = 0.$ (2)

Now, (1) can be regarded as a one parameter family of curves with $a$ as a parameter if $b$ is considered as a function of $a$ given by (2).

Differentiating (1) and (2) w. r. t. $\alpha$ we get,

$f_a + f_b\left(\frac{db}{da}\right) = 0$ (3)

$\therefore g_a + g_b\left(\frac{db}{da}\right) = 0.$

(4)

Comparing (3) and (4) we get $\frac{f_a}{g_a} = \frac{f_b}{g_b} = \lambda$ (say)

$\therefore f_a = \lambda\, g_a$ and $f_b = \lambda\, g_b$.

(5)

Eliminating $a, b$ and $\lambda$ from (1), (2) and (5) we get the required envelope.

**Theorem 2.2**

The envelope touches each member of the given family of curves at the corresponding points.

**Proof**

Let the equation of the one parameter family of curves be

$f(x, y, \alpha) = 0.$ (1)

We know that the envelope is obtained by eliminating $\alpha$ between (1) and

$f_\alpha(x, y, \alpha) = 0.$ (2)

Now, differentiating (1) w. r. t. $x$ we get,

$$f_x(x, y, \alpha) + f_y(x, y, \alpha)\frac{dy}{dx} = 0.$$

$$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y}.$$

$\therefore$ The slope of the tangent to the curve (1) is $-\frac{f_x}{f_y}$.

Now, the envelope can be represented by (1) provided $\alpha$ is regarded as a function of $x, y$ given by (2).

$\therefore$ Differentiating (1) w. r. t. $x$ considering $\alpha$ as a function of $x$ and $y$, we get

$$f_x + f_y\frac{dy}{dx} + f_\alpha\frac{\partial\alpha}{\partial x} + f_\alpha\frac{\partial\alpha}{\partial x}\frac{dy}{dx} = 0.$$

$\therefore f_x + f_y\frac{dy}{dx} = 0.$ (by (2))

$\therefore$ Slope of the tangent to the envelope is $-\frac{f_x}{f_y}$.

$\therefore$ The slopes of the tangents to the curve and the envelope at the common point are equal. Hence they touch each other.

**Note**

In page 99, we have proved that the normals to a curve are tangent to its evolute. Hence the evolute of a curve is the envelope of the normals to the curve.

**Example 25**

Find the envelope of the family of lines $y = mx + \frac{a}{m}$ where $a$ is a constant.

**Solution**

Let $f(x, y, m) = y - mx - \frac{a}{m} = 0$ (1)

$\frac{\partial f}{\partial m} = 0 \Rightarrow -x + \frac{a}{m^2} = 0.$

$\therefore m^2 = \frac{a}{x}.$

(2)

The envelope is got by eliminating between (1) and (2).

From (1) $y^2 = \left(mx + \frac{a}{m}\right)^2.$

$= m^2x^2 + 2ax + \frac{a^2}{m^2}.$

$= ax + 2ax + ax$ [ using (2)]

$= 4ax.$

$\therefore$ The envelope is the parabola $y^2 = 4ax$.

**Example 26**

Find the envelope of the family of curves $\frac{x\cos\alpha}{a} + \frac{y\sin\alpha}{b} = 1$ where $\alpha$ is the parameter and $a$ and $b$ are constants.

**Solution**

$f(x, y, \alpha) = \frac{x\cos\alpha}{a} + \frac{y\sin\alpha}{b} - 1 = 0.$

(1)

$\frac{\partial f}{\partial \alpha} = 0 \Rightarrow -\frac{x\sin\alpha}{a} + \frac{y\cos\alpha}{b} = 0.$ (2)

The envelope is got by eliminating $\alpha$ between (1) and (2).

Squaring (1) and (2) and adding we get

$\left(\frac{x\cos\alpha}{a} + \frac{y\sin\alpha}{b}\right)^2 + \left(-\frac{x\sin\alpha}{a} + \frac{y\cos\alpha}{b}\right)^2 = 1.$

$\frac{x^2}{a^2}(\cos^2\alpha + \sin^2\alpha) + \frac{y^2}{b^2}(\sin^2\alpha + \cos^2\alpha) = 1.$

$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Thus the envelope of the given family of curves is an ellipse.

**Example 27**

Find the envelope of the family of circles $x^2 + y^2 - 2ax\cos\theta - 2ay\sin\theta = c^2$ where $\theta$ is the parameter.

**Solution**

$x^2 + y^2 - 2ax\cos\theta - 2ay\sin\theta = c^2$ (1)

Differentiating w. r. t. $\theta$ we get
$-2ax\sin\theta + 2ay\cos\theta = 0$
(2)
Squaring (1) and (2) and adding we get the envelope as
$4a^2(x^2+y^2) = (x^2+y^2-c^2)^2$.

**Example 28**

Find the envelope of the family of ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where the two parameters $a$ and $b$ are connected by the relation $a+b=c$ where $c$ is a constant.

**Solution**

Using the relation $a+b=c$ the given equation can be written as

$$\frac{x^2}{a^2}+\frac{y^2}{(c-a)^2}=1 \qquad (1)$$

which is a one parameter family of curves with $a$ as the parameter.

Differentiating partially w. r. t. $a$ we get $-\frac{2x^2}{a^3}+\frac{2y^2}{(c-a)^3}=0$.

$\therefore \frac{(c-a)^3}{a^3}=\frac{y^2}{x^2}$.

$\therefore \frac{c-a}{a}=\left(\frac{y^2}{x^2}\right)^{\frac{1}{3}}=\frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}$.

$\therefore a=\frac{cx^{\frac{2}{3}}}{x^{\frac{2}{3}}+y^{\frac{2}{3}}}$ and $c-a=\frac{cy^{\frac{2}{3}}}{x^{\frac{2}{3}}+y^{\frac{2}{3}}}$.

Substituting these values in (1) we get $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$ which is the required envelope.

**Example 29**

Considering the evolute of a curve as the envelope of its normals find the evolute of the ellipse.

**Solution**

Let $P(a\cos\theta, b\sin\theta)$ be any point on the ellipse.

The equation of the normal at $P$ is

$$ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2-b^2 \qquad (1)$$

Thus (1) represents the family of normals with $\theta$ as the parameter.

Differentiating partially w. r. t. $\theta$ we get

$ax\sec\theta\tan\theta + by\,\mathrm{cosec}\,\theta\cot\theta = 0$.

$\therefore\ -\frac{by}{ax} = \frac{\sec\theta\tan\theta}{\operatorname{cosec}\theta\cot\theta} = \tan^3\theta.$

$\therefore \tan\theta = -\left(\frac{by}{ax}\right)^{\frac{1}{3}}.$

Hence $\sin\theta = \pm\frac{(by)^{\frac{1}{3}}}{\left[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}\right]^{\frac{1}{2}}}$ and $\cos\theta = \pm\frac{(ax)^{\frac{1}{3}}}{\left[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}\right]^{\frac{1}{2}}}$.

Substituting in (1) and simplifying we get the equation of the required evolute as $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$.

**Exercise 6**

1. Find the envelopes of the following family of curves.
   (i) $y = mx + \sqrt{a^2 + m^2}$; $m$ is a parameter.
   (ii) $y^2 = 4m(x - m)$; $m$ is a parameter.
   (iii) $x + y\sin\theta = a\cos\theta$; $\theta$ is a parameter.
2. Find the envelope of the family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$ where $a$ and $b$ are connected by the relation.
   (i) $a + b = c$ (ii) $ab = c^2$ (iii) $a^m + b^m = k^m$
3. Considering the evolute of a curve as the envelope of its normals find the evloutes of the following curves.
   (i) $y^2 = 4ax$ (ii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
   (iii) $x = a(\cos\theta + \theta\sin\theta)$; $y = a(\sin\theta - \theta\cos\theta)$.

**Answers**

1. (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ii) $y = \pm x$ (iii) $x^2 - y^2 = a^2$
2. (i) $\sqrt{x} + \sqrt{y} = \sqrt{c}$ (ii) $4xy = c^2$ (iii) $\frac{x^m}{m+1} + \frac{y^m}{m+1} = \frac{k^m}{m+1}$
3. (i) $4(x - 2a^3) = 27ay^2$ (ii) $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}$
   (iii) $x^2 + y^2 = a^2$.

## UNIT-III

## APPLICATION OF DIFFERENTIATION II

### 3.0 Introduction

In this chapter also we shall discuss some important applications of differentiation such as the Maxima and Minima of functions of two variables, errors and approximation, Jacobians, Multiple point, asymptotes, curve tracing and Taylor's series expansion.

### 3.1 Maxima and Minima of function of two variables

The reader is familiar with the method of obtaining the maxima and minima of functions of one variable. It may be recalled that the function $y = f(x)$ has a maximum at $a$ if $f'(a) = 0$ and $f''(a) < 0$ and has a minimum at $a$ if $f'(a) = 0$ and $f''(a) < 0$.

In this section we describe the method of finding the maxima and minima of functions of two variables.

A function $f(x, y)$ of two independent variables $x$ and $y$ is said to have a maximum at $(a, b)$ if

$f(a + h, b + k) - f(a, b) < 0$

for all sufficiently small values of $h$ and $k$.

Suppose $f(x, y)$ attains a maximum or minimum at $(a, b)$. Then $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ at $(a, b)$.

**Working rule for finding maxima and minima of $f(x, y)$.**

**Step 1:** Let $(a, b)$ denote the solution of the equations

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 0 \\ and\ \frac{\partial f}{\partial y} &= 0 \end{aligned}\right\} \qquad (1)$$

**Step 2:** Let $A = \frac{\partial^2 f}{\partial x^2}, B = \frac{\partial^2 f}{\partial x \partial y}$ and $C = \frac{\partial^2 f}{\partial y^2}$ at $(a, b)$.

**Step 3:** Then,

(a) If $AC - B^2 > 0$ and $< 0$ $(or\ B < 0)$, then $f(x, y)$ has a maximum at $(a, b)$.

(b) If $AC - B^2 > 0$ and $> 0$ $(or\ B > 0)$, then $f(x, y)$ has a minimum at $(a, b)$.

(c) If $AC - B^2 < 0$, then $f(x, y)$ has neither a maximum nor a minimum at $(a, b)$. In this case $(a, b)$ is called a saddle point.

(d) No information is obtained if $AC - B^2 = 0$. In such a case further investigation is necessary.

**Example 1**

Find the maximum or minimum values of $u = x^3y^2(1 - x - y)$.

**Solution**

$u = x^3y^2(1 - x - y)$

$\therefore \frac{\partial u}{\partial x} = 3x^2y^2(1 - x - y) - x^3y^2$

$\frac{\partial^2 u}{\partial x^2} = 6xy^2(1 - x - y) - 6x^2y^2$

$\frac{\partial u}{\partial y} = 2x^3y(1 - x - y) - x^3y^2$

$\frac{\partial^2 u}{\partial x \partial y} = 6x^2y(1 - x - y) - 2x^3y - 3x^2y^2$

$\frac{\partial^2 u}{\partial y^2} = 2x^3(1 - x - y) - 2x^3y - 2x^3y$

$\frac{\partial u}{\partial x} = 0 \Rightarrow 3x^2y^2(1 - x - y) - x^3y^2 = 0$

$\Rightarrow x^2y^2[3(1 - x - y) - x] = 0$

$\Rightarrow 4x + 3y = 3$ (1)

$\frac{\partial u}{\partial y} = 0 \Rightarrow 2x^3y(1 - x - y) - x^3y^2 = 0$

$\Rightarrow x^3y[2(1 - x - y) - y] = 0$

$\Rightarrow 2x + 3y = 2$ (2)

Solving (1) and (2) we get $x = \frac{1}{2}$ and $y = \frac{1}{3}$

Now, $A = \frac{\partial^2 u}{\partial x^2}\ at\ \left(\frac{1}{2}, \frac{1}{3}\right)$

$= 6xy^2(1 - x - y) - 6x^2y^2\ at\ \left(\frac{1}{2}, \frac{1}{3}\right)$

$= -\frac{1}{9}$

$B = \frac{\partial^2 u}{\partial x \partial y}\ at\ \left(\frac{1}{2}, \frac{1}{3}\right)$

$= 6x^2y(1 - x - y) - 2x^3y - 3x^2y^2\ at\ \left(\frac{1}{2}, \frac{1}{3}\right)$

$= -\frac{1}{12}$

$C = \frac{\partial^2 u}{\partial y^2}\ at\ \left(\frac{1}{2}, \frac{1}{3}\right)$

$= 2x^3(1 - x - y) - 4x^3y\ at\ \left(\frac{1}{2}, \frac{1}{3}\right)$

$= -\frac{1}{8}$

Now, $AC - B^2 = \frac{1}{72} - \frac{1}{144} = \frac{1}{144}$ is positive.

Also A is negative.

$\therefore$ The function has a maximum at $\left(\frac{1}{2}, \frac{1}{3}\right)$.

The maximum value of u is given by

$u\left(\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}$.

**Example 2**

Prove that $u = x^3 + y^3 - 3axy$ is a maximum or minimum at $x = y = a$ according as $a$ is negative or positive.

**Solution**

$u = x^3 + y^3 - 3axy$.

$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3ay$.

$\frac{\partial^2 u}{\partial x^2} = 6x$.

$\frac{\partial u}{\partial y} = 3y^2 - 3ax$.

$\frac{\partial^2 u}{\partial x \partial y} = -3a$.

and $\frac{\partial^2 u}{\partial y^2} = 6y$.

At $x = y = a$, $A = \frac{\partial^2 u}{\partial x^2} = 6a$.

$B = \frac{\partial^2 u}{\partial x \partial y} = -3a$.

$C = \frac{\partial^2 u}{\partial y^2} = 6a$.

Now, $AC - B^2 = 36a^2 - 9a^2 = 27a^2$ which is positive.

Also, $A = 6a$ is positive or negative according as $a$ is positive or negative.

Hence $u(x, y)$ is maximum or minimum at $x = y = a$ according as $a$ is negative or positive.

**Example 3**

Find the extreme values of $xy(a - x - y)$.

Solution

Let $u = xy(a - x - y)$.

$\therefore \frac{\partial u}{\partial x} = ay - 2xy - y^2.$

$\frac{\partial^2 u}{\partial x^2} = -2y.$

$\frac{\partial u}{\partial y} = ax - x^2 - 2xy.$

$\frac{\partial^2 u}{\partial x \partial y} = a - 2x - 2y.$

$\frac{\partial^2 u}{\partial y^2} = -2x.$

Now, $\frac{\partial u}{\partial x} = 0 \Rightarrow y(a - 2x - y) = 0.$

$\Rightarrow y = 0 \ or \ a - 2x - y = 0.$

(1)

$\frac{\partial u}{\partial y} = 0 \Rightarrow x(a - 2y - x) = 0.$

$\Rightarrow x = 0 \ or \ a - 2y - x = 0.$

(2)

From (1) and (2) we have the following four pairs of equations:

(i) $y = 0, x = 0.$

(ii) $y = 0, a - 2y - x = 0.$

(iii) $a - 2x - y = 0, x = 0.$

(iv) $a - 2x - y = 0, a - 2y - x = 0.$

Solving these equations we get the following four points as solutions

$(0,0), (0, a), (a, 0)$ and $\left(\frac{a}{3}, \frac{a}{3}\right).$

**Case (i):**

At point $(0,0)$ $A = \frac{\partial^2 u}{\partial x^2} = 0;\ B = \frac{\partial^2 u}{\partial x \partial y} = a;\ C = \frac{\partial^2 u}{\partial y^2} = 0.$

$\therefore AC - B^2 = -a^2$, which is negative.

$\therefore u(x, y)$ has neither a maximum nor a minimum at $(0,0)$.

**Case (ii):**

At point $(0, a)$ $A = \frac{\partial^2 u}{\partial x^2} = -2a;\ B = \frac{\partial^2 u}{\partial x \partial y} = -a;\ C = \frac{\partial^2 u}{\partial y^2} = 0.$

$\therefore AC - B^2 = -a^2$, which is negative.

$\therefore u(x, y)$ has neither a maximum nor a minimum at $(0, a)$.

**Case (iii):**

At point $(a, 0)$ $A = \frac{\partial^2 u}{\partial x^2} = 0;\ B = \frac{\partial^2 u}{\partial x \partial y} = -a;\ C = \frac{\partial^2 u}{\partial y^2} = -2a.$

$\therefore AC - B^2 = -a^2$, which is negative.

$\therefore u(x, y)$ has neither a maximum nor a minimum at $(a, 0)$.

**Case (iv):**

At point $\left(\frac{a}{3},\frac{a}{3}\right)$ $A=\frac{\partial^2 u}{\partial x^2}=-\frac{2a}{3};\ B=\frac{\partial^2 u}{\partial x\partial y}=\frac{-a}{3};\ C=\frac{\partial^2 u}{\partial y^2}=-\frac{2a}{3}$.

$\therefore AC-B^2=\frac{4a^2}{9}-\frac{a^2}{9}=\frac{a^2}{3}$, which is positive. Further $A$ is negative or positive according as $a$ is positive or negative.

$\therefore$ At $\left(\frac{a}{3},\frac{a}{3}\right)$, $u(x,y)$ is a maximum or a minimum according as $a>0$ $or$ $a<0$.

The extreme value $= u\left(\frac{a}{3},\frac{a}{3}\right)=\frac{a^2}{9}\left(a-\frac{2a}{3}\right)=\frac{a^2}{27}$.

**Example 4**

Discuss the maxima and minima of $u(x,y)=\sin x\sin y\sin(x+y)$, where $0<x<\pi$ and $0<y<\pi$.

**Solution**

$u(x,y)=\sin x\sin y\sin(x+y)$.

$\therefore \frac{\partial u}{\partial x}=\sin y\,[\sin x\cos(x+y)+\cos x\sin(x+y)]$.

$=\sin y\sin(2x+y)$.

$\frac{\partial^2 u}{\partial x^2}=2\sin y\cos(2x+y)$.

$\frac{\partial u}{\partial y}=\sin x\,[\sin y\cos(x+y)+\cos y\sin(x+y)]$.

$=\sin x\sin(x+2y)$.

$\frac{\partial^2 u}{\partial x\partial y}=\sin x\cos(x+2y)+\cos x\sin(x+2y)$.

$\frac{\partial^2 u}{\partial y^2}=2\sin x\cos(x+2y)$.

Now, $\frac{\partial u}{\partial x}=0\Rightarrow \sin y\sin(2x+y)=0$.

When $0<y<\pi, \sin y\neq 0$.

Hence $\sin(x+2y)=0$. (1)

Similarly, $\frac{\partial u}{\partial y}=0\Rightarrow \sin y\sin(x+2y)=0$. (2)

$\therefore$ From (1) and (2) we get, $2x+y=\pi$ and $x+2y=\pi$.

Solving these equations we get $=\frac{\pi}{3}, y=\frac{\pi}{3}$.

Now, $A=\frac{\partial^2 u}{\partial x^2}$ at $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$

$=\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{2\pi}{3}+\frac{\pi}{3}\right)$

$=2\sin\left(\frac{\pi}{3}\right)\cos\pi=-\sqrt{3}$.

$B = \frac{\partial^2 u}{\partial x \partial y}$ at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$= \sin\left(\frac{4\pi}{3}\right) = -\sqrt{3}/2$ .

$C = \frac{\partial^2 u}{\partial y^2}$ at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$= 2\sin\left(\frac{\pi}{3}\right)\cos\pi = -\sqrt{3}.$

$\therefore AC - B^2 = 3 - \frac{3}{4} = \frac{9}{4}$, which is positive and $A$ is negative.

$\therefore$ $u(x, y)$ is a maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

Maximum value $= \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right) = \frac{3\sqrt{3}}{8}$.

**Exercise 1**

1. Find the maxima and minima for the following functions.
   (i) $x^3 - y^2 - 3x$
   (ii) $2a^2xy - 3ax^2y - ay^3 + x^3y + xy^3$.
   (iii) $xy^2z^3 - x^2y^2z^3 - xy^3z^3 - xy^2z^4$.
   (iv) $\frac{x+y}{x^2+2y^2+6}$
   (v) $2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) + \cos(x+y)$
2. Show that the function $u(x, y) = 2x^2y + x^2 - y^2 + 2y$ has no extreme value.
3. Prove that $x^4 + 2x^2y - x^2 + 3y^2$ is minimum when
   $x = \pm\frac{\sqrt{3}}{2}, y = -\frac{1}{4}$.
4. Prove that $u = x^2y^2 - 5x^2 - 8xy - 5y^2$ is maximum at $x = y = 0$.

**Answers**

1. (i) $x = -1, y = 0$ gives a maximum. Maximum value = 2.
   (ii) Max at $\left(\frac{3a}{2}, -\frac{a}{2}\right), \left(\frac{a}{2}, \frac{a}{2}\right)$; Min at $\left(\frac{a}{2}, -\frac{a}{2}\right)$; neither max or min at $(a, a), (a, -a)$.
   (iii) Max value $\frac{108\, a^7}{7^7}$.
   (iv) Max at $(2, 1)$; Min at $(-2, -1)$.
   (v) Min at $x = y = 2n\pi - \frac{\pi}{2}$; Max at $x = y = n\pi + (-1)^n\ \frac{\pi}{6}$.

## 3.2 Errors and Approximation

If a quantity $u$ is a function of some variables say $x_1, x_2, \dots, x_n$ then an error in one or more of the variables $x_1, x_2, \dots, x_n$ will produce a corresponding error in the value of $u$ also. An important practical application of differential calculus is to determine the error upon the result of the calculations which arise due to the errors in the measurements of quantities on which the calculation depends.

First we shall deal with functions of a single variable.

### 3.2.1 Approximation in the case of functions of one variable

Let $y = f(x)$ be a function having continuous first order differential coefficient. Let there be an error $\Delta x$ in determining the value of $x$. Then the error in the value of $y$ is given by

$\Delta y = f(x + \Delta x) - f(x)$.

$\therefore \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$.

Now, by the definition of derivative

$\frac{\Delta y}{\Delta x} = f'(x) + \varepsilon$ provided $0 < |\Delta x| < \delta$ where $\varepsilon > 0$ is arbitrary.

$\therefore f(x + \Delta x) - f(x) = f'(x)\Delta x + \varepsilon \Delta x$ and $\varepsilon \to 0$ as $\Delta s \to 0$.

$\therefore \Delta y = f'(x)\Delta x$ (approximately).

$\Delta x$ is called the absolute error in $x$.

$\frac{\Delta x}{x}$ is called the relative error in $x$.

$\frac{\Delta x}{x} \times 100$ is called the percentage error in $x$.

For example, suppose that, in measuring the length of a rod of length 10 c. m. an error of 0.2 c. m. is made. Then the absolute error is 0.2 c. m, the relative error is 0.02 c. m. and the percentage error is 2.

**Example 5**

The time of oscillation $T$ of a simple pendulum of length $l$ is given by the formula $T = 2\pi\sqrt{\left(\frac{l}{g}\right)}$. Find the percentage error in $T$ (i) if $l$ is increased by 1% (ii) if the pendulum is removed to a place where $g$ is diminished by 0.04% $l$ remaining unaltered.

**Solution**

We have $T = 2\pi\sqrt{\left(\frac{l}{g}\right)}$ (1)

(i) Taking logarithm and differentiating w. r. t. $l$ we get $\frac{1}{T}\left(\frac{dT}{dl}\right) = \frac{1}{2l}$.

∴ The error relation is $\frac{\Delta T}{T} = \frac{1}{2l}(\Delta l)$.

$$\therefore \left(\frac{\Delta T}{T}\right)100 = \frac{1}{2}\left(\frac{\Delta l}{l}\right)100$$

$$= \frac{1}{2} \quad \text{(since percentage error in } l = 1\text{)}$$

∴ Percentage error in $= \frac{1}{2}$.

(ii) In (1) taking logarithm and differentiating w. r. t. $g$

$$\frac{1}{T}\left(\frac{dT}{dg}\right) = -\frac{1}{2g}.$$

$$\therefore \frac{\Delta T}{T} = -\frac{1}{2}\left(\frac{\Delta g}{g}\right).$$

$$\therefore \left(\frac{\Delta T}{T}\right)100 = -\frac{1}{2}\left(\frac{\Delta g}{g}\right)100$$

$$= \left(-\frac{1}{2}\right)(0.04) = -(0.02).$$

∴ Percentage error in $T = -0.02$.

∴ $T$ is decreased by 0.02%.

**Example 6**

The radius of a sphere is measured and it is 18 c. m. If an error of 0.08 c. m. is made in the value of radius find the percentage error in the volume of the sphere.

**Solution**

Let $r$ be the radius of the sphere.

∴ Volume of the sphere $V = \frac{4}{3}\pi r^3$.

Taking logarithm and differentiating w. r. t. $r$.

$$\frac{1}{V}\left(\frac{dV}{dr}\right) = \frac{3}{r}.$$

∴ The error relation is $\frac{\Delta V}{V} = 3\left(\frac{\Delta r}{r}\right)$.

$$\therefore \left(\frac{\Delta V}{V}\right)100 = 3\left(\frac{\Delta r}{r}\right)100 = 3\left(\frac{0.08}{18}\right)100 = \frac{4}{3}.$$

∴ Percentage error in the volume $= \frac{4}{3}$.

**Example 7**

The area of a triangle is calculated from the angle $A$ and $C$ and the side $b$. If a small error $\delta A$ is made in measuring A show that the percentage error in the area is $\left(\frac{100 \sin C}{\sin A \sin(A+C)}\right)\delta A$ approximately.

**Solution**

Let $S$ be the area of the $\Delta ABC$.

Then $S = \frac{1}{2} bc \sin A$

$$= \frac{1}{2} b \left(\frac{b \sin C}{\sin B}\right) \sin A \qquad \left(\text{since } \frac{b}{\sin B} = \frac{c}{\sin C}\right)$$

$$= \left(\frac{b^2}{2}\right) \frac{\sin A \sin C}{\sin(A+C)}$$

Taking logarithm and differentiating w. r. t. $A$,

$$\frac{1}{S}\left(\frac{dS}{dA}\right) = \frac{\cos A}{\sin A} - \frac{\cos(A+C)}{\sin(A+C)}.$$

$\therefore$ The error relation is $\frac{\Delta S}{S} = \left[\frac{\cos A}{\sin A} - \frac{\cos(A+C)}{\sin(A+C)}\right] \Delta A.$

$$= \frac{\sin C}{\sin A \sin(A+C)} \ \delta A.$$

$\therefore \left(\frac{\Delta S}{S}\right) 100 = \left(\frac{100 \sin C}{\sin A \sin(A+C)}\right) \delta A$ which is the percentage error in $S$.

**Exercise 2**

1. The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of atmost 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

2. The time of swing T of a pendulum is given by $T = k\sqrt{l}$ where $k$ is a constant. Determine the percentage error in the time of swing if the length of the pendulum $l$ changes from 32.1 cm to 32.0 cm.

3. A circular template has a radius of 10 cm $(\pm 0.02)$. Determine the possible error in calculating the area of the templates. Find also the percentage error.

### 3.2.2 Approximations in the case of functions of several variables

**Total differential**

Let $z = f(x, y)$ be a function of two variables with continuous first order partial derivatives.

Then $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$ is called the total increment of $z$. Now ,

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) + f(x, y + \Delta y) - f(x, y).$$

$$= f_x(x + \theta_1 \Delta x, y + \Delta y)\Delta x + f_y(x, y + \theta_2 \Delta y)\Delta y$$

where $0 < \theta_1 < 1; 0 < \theta_2 < 1$.

Now, since the partial derivatives $f_x$ and $f_y$ are continuous,
$\lim_{\Delta x,\Delta y\to 0} f_x(x+\theta_1\Delta x, y+\Delta y) = f_x(x,y)$ and
$\lim_{\Delta x,\Delta y\to 0} f_y(x, y+\theta_2\Delta y) = f_y(x,y)$.
$\therefore f_x(x+\theta_1\Delta x, y+\Delta y) = f_x(x,y)+\varepsilon_1$ and
$f_y(x, y+\theta_2\Delta y) = f_y(x,y)+\varepsilon_2$ where $\varepsilon_1$ and $\varepsilon_2 \to 0$ as $\Delta x$ and $\Delta y \to 0$.
$\therefore\ \Delta z = f_x(x,y)\Delta x + f_y(x,y)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$ (1)
Now, let $\Delta\rho = [(\Delta x)^2 + (\Delta y)^2]^{1/2}$.
Then $\varepsilon_1 \frac{\Delta x}{\Delta\rho}$ and $\varepsilon_2 \frac{\Delta y}{\Delta\rho} \to 0$ as $\Delta\rho \to 0$.
$\therefore \varepsilon_1\Delta x + \varepsilon_2\Delta y$ is an infinitesimal of higher order relative to $\Delta\rho$.

Now, in (i) $f_x(x,y)\Delta x + f_y(x,y)\Delta y$ is linear in $\Delta x$ and $\Delta y$ arc is called the principal part of the increment $\Delta z$ and it differs from $\Delta z$ by an infinitesimal of higher order relative to $\Delta\rho$. The expression $dz = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$ is called the total differential of $f(x,y)$.

$\therefore \Delta z = dz + \varepsilon_1\Delta x + \varepsilon_2\Delta y$ and $\Delta z$ differs from $dz$ by an infinitesimal of higher order relative to $\Delta\rho$. Hence $dz = \Delta z$ approximately.

The increments $\Delta x$ and $\Delta y$ of the independent variables $x$ and $y$ are called the differentials and we denote them by $dx$ and $dy$ respectively. Then the total differential takes the form $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$.

**Note**

1. This idea of total differential can be generalized to functions of several variables.

2. Since the total differential $dz$ is approximately equal to the total increment $\Delta z$, we can calculate the error in $z$ due to the error in the independent variables.

**Example 8**

The range $R$ of a projectile which starts with a velocity $v$ at an elevation $\alpha$ is given by $R = (v^2 \sin 2\alpha)/g$. Find the percentage error in $R$ due to an error of 1% in $v$ and error of $\frac{1}{2}$% in $\alpha$.

**Solution**

We have $R = (v^2 \sin 2\alpha)/g$.

Taking logarithm and differentiating totally,

$\frac{dR}{R} = 2\left(\frac{dv}{v}\right) + 2\cot 2\alpha \ d\alpha.$

$\therefore$ The error relation is $\frac{\Delta R}{R} = 2\left(\frac{\Delta v}{v}\right) + 2\cot 2\alpha \ \Delta\alpha.$

$$\therefore \left(\frac{\Delta R}{R}\right)100 = 2\left(\frac{\Delta v}{v}\right)100 + 2\alpha\cot 2\alpha \left(\frac{\Delta \alpha}{\alpha}\right)100.$$

$$= 2(1) + 2\alpha\cot 2\alpha\left(\frac{1}{2}\right).$$

$$= 2 + \alpha\cot 2\alpha.$$

The percentage error in $R$ is $2 + \alpha\cot 2\alpha$.

**Example 9**

The focal length of a mirror is given by the formula $\frac{1}{v} - \frac{1}{u} = \frac{2}{f}$. If equal errors $\delta$ are made in the determination of $u$ and $v$, show that the relative error in the focal length is given by $\left(\frac{1}{u} + \frac{1}{v}\right)\delta$.

**Solution**

$\frac{1}{v} - \frac{1}{u} = \frac{2}{f} \Rightarrow f = \frac{2uv}{u-v}.$

Taking logarithm and differentiating totally we get

$\frac{df}{f} = \frac{du}{u} + \frac{dv}{v} - \frac{du-dv}{u-v}.$

$\therefore$ The error relation is $\frac{\Delta f}{f} = \frac{\Delta u}{u} + \frac{\Delta v}{v} - \frac{\Delta u - \Delta v}{u-v}$

$$= \frac{\delta}{u} + \frac{\delta}{v} - \frac{\delta-\delta}{u-v} = \left(\frac{1}{u} + \frac{1}{v}\right)\delta.$$

Hence the result.

**Example 10**

Find the percentage error in calculating the area of a rectangle when an error of 2% is made in measuring its sides.

**Solution**

Let $a$ and $b$ be the length and breadth of the rectangle.

$\therefore$ Area $S = ab$.

$\therefore \log S = \log a + \log b$.

Differentiating we get $\frac{dS}{S} = \frac{da}{a} + \frac{db}{b}$.

$$\therefore \left(\frac{\Delta S}{S}\right)100 = \left(\frac{\Delta a}{a}\right)100 + \left(\frac{\Delta b}{b}\right)100 = 2 + 2 = 4.$$

$\therefore$ 4% error is made in calculating the area when there is an error of 2% in measuring the sides.

**Example 11**

A triangle $ABC$ is inscribed in a fixed circle. If the vertices be moved slightly on the circle prove that $\frac{\Delta a}{\cos A}+\frac{\Delta b}{\cos B}+\frac{\Delta c}{\cos C}=0$ where $\Delta a$ is absolute error in $a$ etc.

**Solution**

Since $\Delta ABC$ inscribed in a fixed circle we have $a=2R\sin A$ ; $b=2R\sin B$ and $c=2R\sin C$ where $R$ is the circum-radius of the triangle $ABC$.

Thus the error relations are

$$\Delta a=2R\cos A\,(\Delta A).$$
$$\Delta b=2R\cos B\,(\Delta B).$$
$$\Delta c=2R\cos C\,(\Delta C).$$

$$\therefore \frac{\Delta a}{\cos A}+\frac{\Delta b}{\cos B}+\frac{\Delta c}{\cos C}=2R(\Delta A+\Delta B+\Delta C).$$
$$=2R\,\Delta(A+B+C).$$
$$=2R(\Delta\pi).$$
$$=0 \text{ (since } \pi \text{ is constant).}$$

$$\therefore \frac{\Delta a}{\cos A}+\frac{\Delta b}{\cos B}+\frac{\Delta c}{\cos C}=0.$$

**Example 12**

In a triangle $ABC$ the angles and the sides $b$ and $c$ are made to vary in such a way that the area remains constant. Show that if $b$ and $c$ vary by small amounts $\delta b$ and $\delta c$ respectively then $\delta b\cos B+\delta c\cos C=0$.

**Solution**

In any triangle we have the relation $S=\frac{1}{2}bc\sin A$ where $S$ is the area of the triangle $ABC$.

Taking logarithm and differentiating we get $\frac{dS}{S}=\left(\frac{db}{b}\right)+\left(\frac{dc}{c}\right)+\cot A\ dA$.

Error relation is $\frac{\Delta S}{S}=\frac{\Delta b}{b}+\frac{\Delta c}{c}+\cot A\ \Delta A$.

Since, by hypothesis, $S$ is constant $\Rightarrow \Delta S=0$ we have

$$0=\frac{\delta b}{b}+\frac{\delta c}{c}+\cot A\,(-\Delta B-\Delta C) \qquad (1)$$

(since $A+B+C=\pi \Rightarrow \Delta A+\Delta B+\Delta C=0$)

Also $S=\frac{1}{2}ac\sin B$.

$$\therefore \frac{\Delta S}{S}=\frac{\Delta c}{c}+\cot B\ \Delta B.$$

$$0=\frac{\delta c}{c}+\cot B\,\Delta B.$$

$\therefore \Delta B = -\frac{\delta c}{c}\tan B.$

Similarly, $\Delta C = -\frac{\delta b}{b}\tan C.$

From (1)

$$0 = \frac{\Delta b}{b} + \frac{\Delta c}{c} + \cot A\left(\frac{\delta c}{c}\tan B + \frac{\delta b}{b}\tan C\right).$$

$$= \frac{\delta b}{b}(1 + \cot A \tan C) + \frac{\delta c}{c}(1 + \cot A \tan B).$$

$$= \frac{\delta b}{b}\left(1 + \frac{\cos A \sin C}{\sin A \cos C}\right) + \frac{\delta c}{c}\left(1 + \frac{\cos A \sin B}{\sin A \cos B}\right).$$

$$= \left(\frac{\delta b}{b}\right)\frac{\sin(A+C)}{\sin A \cos C} + \left(\frac{\delta c}{c}\right)\frac{\sin(A+B)}{\sin A \cos B}.$$

$$= \left(\frac{\delta b}{b}\right)\frac{\sin B}{\sin A \cos C} + \left(\frac{\delta c}{c}\right)\frac{\sin C}{\sin A \cos B}.$$

$$= \frac{\delta b}{\cos C} + \frac{\delta c}{\cos B} \quad . \qquad \text{(since } \frac{\sin B}{b} = \frac{\sin C}{c}\text{)}$$

$\therefore \delta b \cos B + \delta c \cos C = 0.$

**Exercise 3**

1. The work that must be done to propel a ship of displacement $D$ for a distance $s$ in time $t$ is proportional to $\left(\frac{s^2 D^{\frac{3}{2}}}{t^2}\right)$. Estimate roughly the percentage increase or decrease of work necessary when the distance is increased by 1% the time is diminished by 1% and the displacement of the ship is diminished by 3%.

2. The side $a$ of a triangle $ABC$ is calculated from the sides $b$ and $c$ and angle $A$. If small errors $\delta c, \delta b, \delta A$ are made in the values of $c, b$ and $A$ respectively prove that the error $\delta a$ in the calculated value of $a$ is equal to $\cos B \;\; \delta c + \cos C \;\; \delta b + b \sin C \;\; \delta A$.

## 3.3 Jacobians

In this section we introduce the concept of Jacobian of a transformation which plays a vital role in change of variables in any transformation of one coordinate system to another. This concept is useful in any transformation of one coordinate system to another. This concept is useful in the theory of integral calculus in the evaluation of double and triple integrals.

**Definition**

Consider the transformation given by $x = x(u, v), y = y(u, v)$ where the function $x$ and $y$ have continuous first order partial derivatives. Then the Jacobian of the transformation is defined as

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

The Jacobian $J$ is also denoted by $= \frac{\partial(x,y)}{\partial(u,v)}$.

Hence $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$.

For a transformation in three variables

$x = x(u, v, w), y = y(u, v, w), z = z(u, v, w)$ the Jacobian $J$ is given by the following determinant of order three.

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

**Example 13**

The transformation from Cartesian coordinates $(x, y)$ to polar coordinates $(r, \theta)$ is given by $x = r\cos\theta$ and $y = r\sin\theta$.

**Solution**

$$\therefore J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

**Example 14**

The transformation from Cartesian coordinates $(x, y, z)$ to cylindrical polar coordinates $(r, \theta, z)$ is given by $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

Then $J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$

**Example 15**

The transformation from Cartesian coordinates $(x, y, z)$ to cylindrical polar coordinates $(r, \theta, \phi)$ is given by $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$.

Then $J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$

$$= \sin\theta\cos\phi\,(r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi\,(-r\sin\theta\cos\theta\cos\phi)$$
$$- r\sin\theta\sin\phi\,[-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi]$$

$= r^2[\sin^3\theta\,(\cos^2\phi + \sin^2\phi) + \cos^2\theta\sin\theta\,(\cos^2\phi + \sin^2\phi)]$

$= r^2(\sin^3\theta + \cos^2\theta\sin\theta)$

$= r^2\sin\theta\,(\sin^2\theta + \cos^2\theta) = r^2\sin\theta.$

**Example 16**

Consider the transformation $x + y = u, 2x - 3y = v$.

Solving the two equations we get $x = \frac{3}{5}u + \frac{1}{5}v$ and $y = \frac{2}{5}u - \frac{1}{5}v$.

$$\therefore J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{vmatrix} = -\frac{1}{5}$$

**Note** Hence $u$ and $v$ are functions of $x$ and $y$. The Jacobian $J'$ of $u$ and $v$ w. r. t. $x$ and $y$ can be written as $J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5.$

**Properties of Jacobians**

**Result 1**

Let $x = x(u, v); y = y(u, v)$. If $J = \frac{\partial(x,y)}{\partial(u,v)}$ and $J' = \frac{\partial(u,v)}{\partial(x,y)}$ then $JJ' = 1$.

**Proof**

Since $x = x(u, v)$ and $y = y(u, v)$ we shall assume that we can solve these equations to obtain the values of $u$ and $v$ in terms of $x$ and $y$. Hence we have $u = u(x, y)$ and $= v(x, y)$ .

We have $\frac{\partial x}{\partial x} = 1$; $\frac{\partial y}{\partial y} = 1$; $\frac{\partial x}{\partial y} = 0 = \frac{\partial y}{\partial x}$.

Now, $\frac{\partial x}{\partial x} = 1 \Rightarrow \frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial x} = 1$

$\frac{\partial x}{\partial y} = 0 \Rightarrow \frac{\partial x}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial y} = 0.$

$\frac{\partial y}{\partial x} = 0 \Rightarrow \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} = 0.$

$\frac{\partial y}{\partial y} = 1 \Rightarrow \frac{\partial y}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial y} = 1.$

Now, $JJ' = \frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \times \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$$= \begin{vmatrix} \frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial x}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial y} \\ \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial y}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$\therefore JJ' = 1.$

**Result 2**

If $u$ and $v$ are functions of $r$ and $s$ and $r$ and $s$ are functions of $x, y$ then $\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,y)}$.

**Proof**

$$\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)}.$$

**Example 17**

If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1} x + \tan^{-1} y$ find $\frac{\partial(u,v)}{\partial(x,y)}$.

**Solution**

$\frac{\partial u}{\partial x} = \frac{(1-xy)+(x+y)y}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}.$

Similarly, $\frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}$; $\frac{\partial v}{\partial x} = \frac{1}{1+x^2}$; $\frac{\partial v}{\partial y} = \frac{1}{1+y^2}$.

Now, $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$

$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$

**Example 18**

If $u = 2xy, v = x^2 - y^2, x = r\cos\theta\,, y = r\sin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$.

**Solution**

$u = 2xy = 2r^2\cos\theta\sin\theta.$

$= r^2\sin 2\theta.$

$v = x^2 - y^2 = r^2\cos^2\theta - r^2\sin^2\theta$

$= r^2\cos 2\theta.$

Now $\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2r\sin 2\theta & 2r^2\cos 2\theta \\ 2r\cos 2\theta & -2r^2\sin 2\theta \end{vmatrix}$

$= -4r^3(\sin^2 2\theta + \cos^2 2\theta).$

$= -4r^3.$

**Example 19**

If $u = x + y + z$ (1)

$uv = y + z$

(2)

$uvw = z$

(3)

Show that $\frac{\partial(x,y,z)}{\partial(u,vw)} = u^2v$ .

**Solution**

First we solve for $x, y, z$ in terms of $u, v, w$.

$(2) \Rightarrow y = uv - z.$

$\Rightarrow y = uv - uvw = uv(1-w).$

$(1) \Rightarrow u = x + (y+z) \Rightarrow u = x + uv$

$\Rightarrow x = u(1-v).$

$(3) \Rightarrow z = uvw.$

Now $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)[u^2v(1-w) + u^2vw] + u[uv^2(1-w) + uv^2w]$$

$$= u^2v - u^2v^2 + u^2v^2.$$

$$= u^2v.$$

**Example 20**

If $x = u\cos v\,, y = u\sin v$ prove that $\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = 1.$

**Solution**

Given $x = u\cos v$

(1)

$y = u\sin v$

(2)

$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} = u.$

Also from (1) and (2) we get $u = \sqrt{x^2+y^2}$ and $v = \tan^{-1}(y/x)$

$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$

$$= \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}}.$$

$$= \frac{x^2+y^2}{(x^2+y^2)^{3/2}}.$$

$$= \frac{1}{\sqrt{x^2+y^2}}.$$

$$= \frac{1}{u}.$$

Now, $\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = u \times \frac{1}{u} = 1.$

**Example 21**

If $u = \frac{yz}{x}, v = \frac{zx}{y}, w = \frac{xy}{z}$ prove that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4.$

**Solution**

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{xyz}\begin{vmatrix} -\frac{yz}{x} & z & y \\ z & -\frac{zx}{y} & x \\ y & x & -\frac{xy}{z} \end{vmatrix}$$

$$= \frac{1}{xyz}\left[-\frac{yz}{x}(x^2 - y^2) - z(-xy - xy) + y(xz + xz)\right]$$

$$= \frac{1}{xyz}[0 + 2xyz + 2xyz] = 4.$$

**Exercise 4**

1. Find the Jacobian of the following transformations.

(i) $2x + 3y = u; x - 2y = v.$ (ii) $u = \frac{x^3}{y}; v = \frac{y^3}{x}$

(iii) $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta.$

(iv) $x = u(1 + v);\ y = v(1 + u)$

2. If $x = e^u\cos v$ and $y = e^u\sin v$ show that $\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = 1.$

3. If $u = x + y; v = x - y$ and $x = r\cos\theta; y = r\sin\theta$ prove that $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(r,\theta)}$

4. If $u = \frac{x}{\sqrt{1-r^2}}; v = \frac{y}{\sqrt{1-r^2}}; w = \frac{z}{\sqrt{1-r^2}}$ where $r^2 = x^2 + y^2 + z^2$ prove that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = (1 - r^2)^{-\frac{5}{2}}.$

5. If $x = e^v\sec u$ and $y = e^v\tan u$, find the Jacobians $\frac{\partial(x,y)}{\partial(u,v)}$ and $\frac{\partial(u,v)}{\partial(x,y)}$. Verify that $\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1.$

**Answers**

1. (i) $-\frac{1}{7}$ (ii) $\frac{1}{8\sqrt{uv}}$ (iii) $-r^2\cos\theta$ (iv) $1 + u + v.$

### 3.4 Multiple point

On some curves there exist some special points through which more than one branch of the curve pass. Such points are called singular points.

In this section we classify various types of singular points lying on a curve.

**Multiple points**

**Definition**

If $r$ branches of a curve pass through a point then that point is called a multiple point of $r^{th}$ order.

A multiple point of order two is called a double point.

**Note** In general there are $r$ tangents one for each branch at a multiple point of order $r$. All the tangents need not be distinct and real. Hence we can classify the multiple points according to the nature of the tangents at the multiple points.

**Classification of double points**

**Definition**

A double point P is called a node if the two tangents at P are real and distinct. In this case two real branches of the curve pass through the point P.

A double point P is called a cusp if the two tangents at P are real and coincident. In this case two real branches of the curve touch at the point P.

If the tangents at P are imaginary then there are no real points on the curve in the immediate neighbourhood of the point. Such a point is called a conjugate point. Hence a conjugate point is an isolated point whose coordinates satisfy the equation of the curve.

If the two branches of the curve at a cusp P lie on the opposite sides of the common tangent then P is called a cusp of the first kind or cusp of the first species.

If the two branches of the curve at a cusp P lie on the same side of the common tangent then P is called a cusp of the second kind or cusp of the second species.

Condition for a point $(x, y)$ to be a multiple point of the curve $f(x, y) = 0$.

Let the curve be $f(x, y) = 0$
(1)

Differentiating (1) with respect to $x$ we get

$$f_x + f_y\left(\frac{dy}{dx}\right) = 0 \qquad (2)$$

Where $f_x$ and $f_y$ denote the partial derivatives with respect to $x$ and $y$ respectively.

At a multiple point the curve has more than one tangent. Hence $\left(\frac{dy}{dx}\right)$ must have more than one value at a multiple point.

But equation (2) is of first degree in $\left(\frac{dy}{dx}\right)$. Hence $\left(\frac{dy}{dx}\right)$ can have more than one value if and only if $f_x = f_y = 0$.

$\therefore$ A point $(x, y)$ on the curve $f(x, y) = 0$ is a multiple point if and only if $f_x = f_y = 0$.

**Working rule**

To find the multiple points on the curve we have to find those values $(x, y)$ which simultaneously satisfy the three equations $f(x, y) = 0; f_x(x, y) = 0; f_y(x, y) = 0.$

**The nature of the double point**

Let $(x, y)$ be a double point on the curve $f(x, y) = 0$.

Then $f_x(x, y) = 0; f_y(x, y) = 0.$

We assume that $f(x, y)$ has continuous partial derivatives of second order and they are not zero.

Now, differentiating (2) with respect to $x$ we get

$$f_{xx} + f_{xy}\left(\frac{dy}{dx}\right) + f_{yx}\left(\frac{dy}{dx}\right) + f_{yy}\left(\frac{dy}{dx}\right)^2 + f_y\left(\frac{d^2y}{dx^2}\right) = 0.$$

$$\therefore f_{xx} + 2f_{xy}\left(\frac{dy}{dx}\right) + f_{yy}\left(\frac{dy}{dx}\right)^2 = 0 \qquad (3)$$

$(\because f_y = 0 \text{ and } f_{xy} = f_{yx})$

Equation (3) is a quadratic in $\frac{dy}{dx}$.

Since $f_{xx}, f_{xy}$ and $f_{yy}$ are not all zero, two roots $\frac{dy}{dx}$ given by (3) will be real and distinct or coincident or imaginary according as the discriminant of the quadratic equation (3) is greater than zero or equal to zero or less than zero.

The discriminant is given by $4\left(f_{xy}\right)^2 - 4f_{xx}f_{yy}$ .

$\therefore$ The point is a node if $\left(f_{xy}\right)^2 - f_{xx}f_{yy} > 0.$

The point is a cusp if $\left(f_{xy}\right)^2 - f_{xx}f_{yy} = 0.$

The point is a conjugate point if $\left(f_{xy}\right)^2 - f_{xx}f_{yy} < 0.$

**Note**

1. The above conditions for the nature of double points are in fact the conditions for two tangents at the double point to be real and distinct or coincident or imaginary. Hence it cannot be always taken as a test for node cusp or conjugate point.

2. The nature of the double point of a curve can be achieved by shifting the origin to the double point and then testing the nature of tangents and existence of the curve in the neighbourhood of the new origin.

[ The equation (equations) of the tangents at the origin is obtained by equating to zero the lowest degree terms in the equation of the curve].

3. If $f_{xx} = f_{xy} = f_{yy} = 0$ then the point $(x, y)$ will be a multiple point of higher order.

**Example 22**

Find the position and nature of the double points of the curve $a^4y^2 = x^4(2x^2 - 3a^2)$.

**Solution**

Let $f(x, y) = 2x^6 - 3a^2x^4 - a^4y^2 = 0$ (1)

$\therefore f_x = 12x^5 - 12a^2x^3$

$f_{xx} = 60x^4 - 36a^2x^2.$

$f_{xy} = 0.$

$f_y = -2a^4y$

$f_{yy} = -2a^4$ .

The double point are got from $f_x = 0$ and $f_y = 0$.

Now $f_x = 0 \Rightarrow 12x^5 - 12a^2x^3 = 0$

$\Rightarrow 12x^3(x^2 - a^2) = 0.$

$\Rightarrow x = 0, a, -a.$

Also, $f_y = 0 \Rightarrow -2a^4y = 0$

$\Rightarrow y = 0.$

Hence the double points are $(0,0), (a,0), (-a,0)$.

Of these three points, only $(0,0)$ lies on the curve.

$\therefore$ $(0,0)$ is the only double point.

At $(0,0)$, $f_{xx} = 0; f_{yy} = -2a^4; f_{xy} = 0.$

Now, $\left(f_{xy}\right)^2 - f_{xx}f_{yy} = 0 - 0 \times (-2a^4) = 0.$

$\therefore$ The double point $(0,0)$ is a cusp.

But form (1), $y = \pm\frac{x^2}{a^2}\sqrt{2x^2 - 3a^2}.$

Hence for small values of $x$, positive or negative, $2x^2 - 3a^2$ is a negative.

Hence $y$ is imaginary.

$\therefore$ No portion of the curve lies in the neighbourhood of the origin.

Hence the origin is a conjugate point but not a cusp.

**Example 23**

Find the position and nature of the double points of the curve $x^3 + x^2 + y^2 - x - 4y + 3 = 0.$

**Solution**

Let $f(x,y) = x^3 + x^2 + y^2 - x - 4y + 3 = 0$ (1)

$\therefore$ $f_x = 3x^2 + 2x - 1; f_{xx} = 6x + 2; f_{xy} = 0$

$f_y = 2y - 4; f_{yy} = 2.$

The double points are got from $f_x = 0$ and $f_y = 0$.

Now $f_x = 0 \Rightarrow 3x^2 + 2x - 1 = 0$

$\Rightarrow (3x - 1)(x + 1) = 0.$

$\Rightarrow x = -1, \frac{1}{3}.$

Now $f_y = 0 \Rightarrow 2y - 4 = 0$

$\Rightarrow y = 2.$

$\therefore$ The possible double points are $(-1,2)$ and $\left(\frac{1}{3}, 2\right)$.

We note that out of the two points only $(-1,2)$ lies on the curve

Hence $(-1,2)$ is the only double point.

At $(-1,2)$, $f_{xy} = 0; f_{xx} = -4; f_{yy} = 2.$

$\therefore$ At $(-1,2)$, $\left(f_{xy}\right)^2 - f_{xx}f_{yy} = 0 - (-4) \times 2 = 8 > 0.$

Hence the double point $(-1,2)$ is a node.

**Example 24**

Find the position and nature of the double points of the curve $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$.

**Solution**

Let $f(x, y) = x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$ (1)

$\therefore f_x = 3x^2 - 6xy - 1; f_y = 3x^2 - 12y^2 + 1.$

The double points are got from $f_x = 0$ and $f_y = 0$.

Now $f_x = 0 \Rightarrow 3x^2 + 6xy - 1 = 0$

(2)

$f_y = 0 \Rightarrow 3x^2 - 12y^2 + 1 = 0$ (3)

From (2) we get $y = \frac{1-3x^2}{6x}$.

Using (2) in (3) we get $3x^2 - 12\left(\frac{1-3x^2}{6x}\right)^2 + 1 = 0$

$\therefore 9x^4 - (1 + 9x^4 - 6x^2) + 3x^2 = 0.$

$\therefore 9x^2 - 1 = 0.$ Hence $= \pm\frac{1}{3}$.

Substituting the values of $x$ in (3) we get $12y^2 = \frac{4}{3}$. Hence $= \pm\frac{1}{3}$.

The possible double points are $\left(\frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, -\frac{1}{3}\right), \left(-\frac{1}{3}, \frac{1}{3}\right), \left(-\frac{1}{3}, -\frac{1}{3}\right)$.

Out of these four points none satisfies the equation (1).

Hence there are no double points for the given curve.

**Exercise 5**

Find the position and nature of the double points on the following curves

(i) $x^2(x - y) + y^2 = 0$ (ii) $x^3 + y^3 - 12x - 27y + 70 = 0$

(iii) $xy^2 - ax^2 + 2a^2x - a^3 = 0$.

**Answers**

(i) cusp at $(0,0)$ (ii) Conjugate point at $(0, 0)$ (iii) node at $(a, 0)$

**Kinds of cusps**

Of the three types of double points the cusp can be distinguished as two special kinds according to the nature of the curve at the cusp.

We know that at a cusp two branches of a curve have a common tangent and hence the have a common normal also.

**Single cusp**

A cusp is said to be a single cusp if the two branches of the curve lie entirely on one side of the common normal at the cusp.

**Double cusp**

A cusp is said to be a double cusp if the two branches of the curve extend to both sides of the common normal at the cusp.

**Cusp of first kind (first species)**

If the branches of the curve lie on opposite sides of the common tangent at the cusp, the cusp is called the cusp of the first kind.

**Cusp of second kind (second species)**

If the branches of the curve lie on the same side of the common tangent at the cusp, the cusp is called the cusp of the second kind.

**Working rule to find the nature of the cusp at the origin**

**Case 1:** The cuspidal tangents are $y^2 = 0$

In this case solve the given equation for $y$ neglecting terms containing powers of $y$ higher than two.

(i) Single cusp: if the roots are real for one sign of $x$.

(ii) Double cusp: if the roots are real for both signs of $x$.

(iii) First species: if the roots are opposite in sign.

(iv) Second species: if the roots are of the same sign.

**Case 2:** The cuspidal tangents are $x^2 = 0$

In this case solve the given equation for $x$ neglecting terms containing powers of $x$ higher than two.

(i) Single cusp: if the roots are real for one sign of $y$.

(ii) Double cusp: if the roots are real for both signs of $y$.

(iii) First species: if the roots are opposite in sign.

(iv) Second species: if the roots are of the same sign.

**Case 3:** The cuspidal tangents are $(ax + by)^2 = 0$.

In this case put $p = ax + by$ and eliminate $y$ or $x$ (whichever is convenient) from the given equation of the curve. Suppose we eliminate $y$ then we get an equation in $p$ and $x$. Solve the equation for $p$ (neglecting $p^3$ and higher power of $p$). Nature of the cusp will be decided as in case 1 (taking $p$ for $y$) or case 2 (taking $p$ for $x$).

**Case 4:** Nature of the cusp at a point other than the origin

Transfer the origin to that point and proceed as in case 1 or case 2 or case 3 as the case may be.

**Example 25**

Show that the curve $y^2(2a-x)=x^3$ has a single cusp of first species at the origin.

**Solution**

The equation of the curve is $x^3+xy^2-2ay^2=0$ (1)

Equating to zero the lowest degree terms we get $-2ay^2=0$.

$\therefore$ $y^2=0$, and it's roots are real and coincident.

Hence the origin is a cusp or a conjugate point.

From (1), we get $= \pm x\sqrt{\frac{x}{2a-x}}$. (2)

When $x$ is small and positive $y$ is real. Hence real branches of the curve pass through the origin.

$\therefore$ The origin is a cusp.

Also from (2), $y$ is real if $x$ is small and positive.

$\therefore$ The cusp is a single cusp.

Also for any small and positive value of $x$ the two values of $y$ are of opposite signs.

$\therefore$ The cusp is of first species.

Hence the origin is a single cusp of first species.

**Example 26**

Show that the curve $y^3=(x-a)^2(2x-a)$ has a single cusp of first species at $(a,0)$.

**Solution**

The equation of the curve is $y^3=(x-a)^2(2x-a)$ (1)

Shifting the origin to the point $(a,0)$ by putting $x=X+a, y=Y$ equation (1) is transformed to $Y^3=X^2(2X-a)$ (2)

Equating to zero the lowest degree terms we get $aX^2=0$, whose roots are real and coincident.

Hence the new origin $(a,0)$ is a cusp or a conjugate point.

From (2) solving for $X$ neglecting $X^3$ and higher powers of $X$ we get

$$X = \pm Y\sqrt{\frac{Y}{a}}$$

(3)

When $Y$ is small and positive $X$ is real. Hence real branches of the curve pass through $(a, 0)$.

Hence $(a, 0)$ is a cusp.

From (3), for one sign (positive) of $Y$ the value of is real.

∴ The cusp is a single cusp.

Also for any small and positive value of $Y$ the two values of $X$ are of opposite sign.

∴ The cusp is of first species.

Hence $(a, 0)$ is a single cusp of first species.

**Exercise 6**

1. Show that the curve $y^2(2a - x) = x^3$ has a single cusp of the first species at the origin.
2. Show that the curve $y^3 = x^3 + ax^2$ has a single cusp of first species at the origin.

## 3.5 Curve Tracing

We are familiar with some standard curves such as the straight line, circle, parabola, ellipse, hyperbola whose equations in standard forms are respectively $y = mx + c; x^2 + y^2 = a^2; y^2 = 4ax; \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

If an equation $f(x, y) = 0$ can be reduced to the above standard forms the curve represented by the equation can easily be traced with their known properties.

In this section we discuss the methods of tracing the curve whose equations are given in terms of Cartesian coordinates, polar coordinates, and parametric coordinates.

The aim of curve tracing is to find an approximate shape of the curve. The knowledge of the nature and shape of the curves are useful when we evaluate length, areas, volumes, surface areas etc of a bounding curve.

**A: Tracing of curves $f(x, y) = 0$ (Cartesian Coordinates)**

Suppose a curve is represented in terms of Cartesian coordinates by the equation $f(x, y) = 0$. The following points provide useful information's regarding the shape and nature of the curve.

**I. Symmetry of the curve**

**(a) Symmetry about the $x$-axis:**

A curve $f(x, y) = 0$ is symmetric about the $x$-axis if $f(x, -y) = f(x, y)$.

**Example** $y^2 = 4ax; x^2 + y^2 = a^2; y^4 + y^2 + x^3 = 0$ are curves which are symmetric about the $x$-axis.
But $x^2 + y^2 = ay$ is not symmetric about the $x$-axis.

**(b) Symmetry about the $y$-axis**

A curve $f(x, y) = 0$ is symmetric about the $y$-axis if $f(-x, y) = f(x, y)$.

**Example** $x^2 = 4ay; x^2 + y^2 = a^2; y = x^4 + x^2 + a$ are symmetric about $y$-axis.
But $x^2 + y^2 = ax$ is not symmetric about the $y$-axis.
**Note** $x^2 + y^2 = a^2$ is symmetric about $x$-axis and $y$-axis. In this case the equation involves even and only even powers of $x$ as well as $y$.

**(c) Symmetry about the line $y = x$.**

If $f(x, y) = f(y, x)$ then the curve is symmetric about the line $y = x$.

**Example** $x^2 + y^2 = a^2; x^3 + y^3 = 3axy; xy = c^2$ are symmetric about the line $y = x$.

**(d) Symmetric about the origin. (Symmetric in opposite quadrants)**

If $f(-x, -y) = f(x, y)$ then the curve is symmetric about the origin (symmetric in opposite quadrants).

**Example** $x^2 + y^2 = a^2; xy = c^2$ are symmetric about the origin.
$x^3 + y^3 = 3axy; y^2 = x^3$ are not symmetric about the origin.

**II. Points of intersection with the coordinate axes**

To obtain the points where the curve $f(x, y) = 0$ intersects the $x$-axis put $y = 0$ in the equation and solve for $x$. Similarly, to find the points where the curve intersects the $y$-axis put $x = 0$ in the equation and solve for $y$.

**Example** The curve $x^2 + y^2 = a^2$ crosses the $x$-axis at $(a, 0)$ and $(-a, 0)$ and crosses the $y$-axis at $(0, a)$ and $(0, -a)$.

The curve $y^2 = 4ax$ passes through the origin.

**III. Region in which the curve lies.**

If the equation of the curve $f(x, y) = 0$ can be expressed in the form $y = g(x)$ we determine the values of $x$ for which $y$ is imaginary or $y$ is not defined. No portion of the curve lies in the corresponding region.

Similar information can be obtained if the equation of the curve can be expressed in the form $x = g(y)$.

**Example** The curve $y^2(a - x) = x^3$ can be written as $= x\sqrt{\frac{x}{a-x}}$. Clearly $y$ is imaginary when $x > a$ or $x < 0$. Hence the curve does not lie on the left of the $y$-axis and to the right of the line $x = a$.

**IV. Tangents to the curve**

**(a) Tangents at the origin**

If the origin is found to be a point on the curve then the tangents at the origin are obtained by equating to zero the lower degree terms occurring in the equation.

**Example** $y^2 = 4ax$ passes through the origin and lower degree term occurring in it is $4ax$ which when equated to zero becomes $4ax = 0$ (i.e.) $x = 0$. Hence $y$-axis is the tangent to the parabola at the origin.

Also $x^3 + y^3 = 3axy$ passes through the origin at which $x = 0$ and $y = 0$ are the tangents.

For the curve $a^2y^2 = a^2x^2 - x^4, y = \pm x$ are the tangents at the origin.

**(b) Tangents at any other point $(h, k)$ other than the origin**

Find $\frac{dy}{dx}$ at $(h, k)$ and it gives the slope of the tangent to the curve at this point. This will be useful to decide the nature of the tangent – whether parallel to the $x$-axis or $y$-axis or inclined tangent.

**V. Asymptotes**

The concept of asymptotes described in the previous chapter will be helpful to know about the asymptotes in tracing any curve.

**(a) Asymptotes parallel to the $x$-axis.**

These are obtained by equating to zero the coefficient of the highest power of $x$.

**Example** $(y + a)x^2 + x - 1 = 0$ has an asymptote $y = -a$ parallel to the $x$-axis.

**(b) Asymptotes parallel to the $y$-axis.**

These are obtained by equating to zero the coefficients of the power of $y$.

**Example** $y^2(4 - x^2) = x^3 - 1$ has asymptotes $4 - x^2 = 0$ (i.e.) $x = 2$ and $x = -2$ are two asymptotes parallel to the $y$-axis.

**(c) Inclined asymptotes**

Taking $y = mx + c$ as an asymptote we can find $m$ and $c$ by substituting $y = mx + c$ in the equation and equating to zero the various powers of $x$ starting from the highest power.

**Example** For the curve $x^3 + y^3 = 3axy; x + y + a = 0$ is an inclined asymptote.

**VI. Special Points**

Points at which the function is maximum or minimum; the points of inflexion; intervals in which the function is increasing or decreasing; region of concavity and convexity; multiple points such as cusp, node, conjugate points provide useful information's in determining the shape of the curve.

Having known all these information's by inspection or investigation we shall trace the curve.

**B: Tracing a curve $f(r, \theta) = 0$ (polar coordinates)**

To trace a curve given in terms of polar coordinates by the equation $f(r, \theta) = 0$ we investigate the following.

**I. Symmetry of the curve**

**(a) Symmetry about the initial line.**

The curve $f(r, \theta) = 0$ is symmetric about the initial line $\theta = 0$ if $f(r, -\theta) = f(r, \theta)$.

**Example** $r = a(1 + \cos\theta)$; $r = a(1 - \cos\theta)$; $r = a\cos 2\theta$ are symmetric about the initial line. However $r = a(1 - \sin\theta)$ is not symmetric about the initial line.

**(b) Symmetry about the pole**

The curve is symmetric about the pole if $f(-r, \theta) = f(r, \theta)$.

**Example** $r^2 = a^2 \cos 2\theta$ ; $r^2 = a^2 \sin 2\theta$ are symmetric about the pole.

**(c) Symmetry about $= \frac{\pi}{2}$.**

The curve $f(r, \theta) = 0$ is symmetric about the line $= \frac{\pi}{2}$. ($y$-axis) if $f(r, \pi - \theta) = f(r, \theta)$.

**Example** $r = a(1 + \sin\theta)$; $r = a\sin 3\theta$ are symmetric about $= \frac{\pi}{2}$.

**II. Tangents at the pole.**

We put $r = 0$ in the equation of the curve and solve the resulting equation for $\theta$. If there exists a real solution $\alpha$ for $\theta$, then the curve passes through the pole and the line $\theta = \alpha$ is a tangent to the curve at the pole.

**III. Region in which the curves lies.**

(i) If the maximum value of $r$ is $a$, then the curve lies within the circle $r = a$.

(ii) If there exist values of $\theta$ for which $r^2 < 0$ so that $r$ becomes imaginary then the curve does not exist for those values of $\theta$.

**Example** $r^2 = a^2 \sin 2\theta$ does not exist if $\frac{\pi}{2} < \theta < \pi$.

**IV. Value of $\phi$.**

The angle $\phi$ which a tangent at $(r, \theta)$ makes with the initial line is found from the formula $\tan\phi = r\frac{d\theta}{dr}$.

**V. Asymptotes**

If there is no finite value $\alpha$ for $\theta$ such that $r \to \infty$, then the curve $f(r, \theta) = 0$ has no asymptotes.

**VI. Points on the curve**

Giving different values for $\theta$ we can get different points on the curve which will be of use in tracing the curve and ascertain whether $r$ increases or decreases in the region.

**C: Tracing a curve $x = f(t), y = g(t)$ (Parametric equations)**

(i) Suppose $x = f(t), y = g(t)$ are parametric equations of a curve where t is the parameter.

If it is possible to eliminate the parameter between the two equations and get the Cartesian form of the curve we proceed as in the case of Cartesian coordinates.

(ii) If the parameter $t$ cannot be easily eliminated

(a) Find $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$.

(b) Give different values to the parameter $t$ and find, , $y, \frac{dy}{dx}$. This gives different points on the curve and slopes of the tangents at these points.

(c) We plot the points and trace the curve.

**Example 27**

Trace the curve $y^2 = ax^3$ (semi cubical parabola)

**Solution**

The curve is symmetric about $x$-axis.

It passes through the origin.

It has a tangent $y = 0$ ($x$-axis) at $(0,0)$.

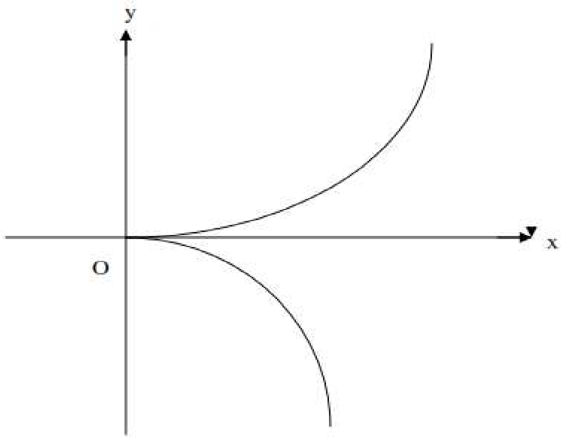
The curve has no asymptotes.

Since $y$ is imaginary when $x < 0$ no part of the curve lies to the left of the $y$-axis.

The curve does not cut the axis except at the origin.

Since the $x$-axis is the tangent and the curve is symmetric about the $x$-axis the two branches of the curve lie on either side of the tangent. Hence origin is a cusp.

The form of the curve is as shown in the figure.




The curve is called semi cubical parabola.

**Example 28**

Trace the conic $r = 2a\cos\theta$ (circle)

**Solution**

The curve is symmetric about the initial line.

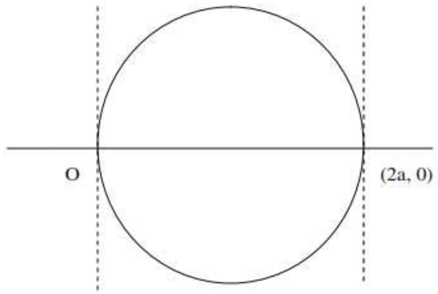
When, $\theta = \frac{\pi}{2}, r = 0$. Hence the curve passes through the pole and further $\theta = \frac{\pi}{2}$ is the tangent at the pole.

We can prove $= \theta + \frac{\pi}{2}$. When $\theta = 0$ we have $r = 2a$ and $= \frac{\pi}{2}$. Hence at $(2a, 0)$ the tangent is perpendicular to the initial line.

Since $|\cos\theta| \leq 1$ we have $r \leq 2a$. Hence the whole portion of the curve lies within the circle $r = 2a$.

Some points on the curve are given in the table.

| $\theta$ | $0$ | $\pi/4$ | $\pi/2$ |
|---|---|---|---|
| $r$ | $2a$ | $\sqrt{2}\,a$ | $0$ |




The form of the curve is as shown in the figure.

**Example 29**

Trace the curve $r = a(1 + \cos\theta)$ where $a > 0$ (cardioid).

**Solution**

We note the following from the equation of the given curve.

The curve is symmetric about the initial line.

When $\theta = \pi$ we have $r = 0$. Hence the curve passes through the pole and further $\theta = \pi$ is the tangent at the pole.

Let $\phi$ be the angle made by the tangent at $(r, \theta)$ with the initial line.

Now, $\tan\phi = r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta} = -\cot\left(\frac{\theta}{2}\right) = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$.

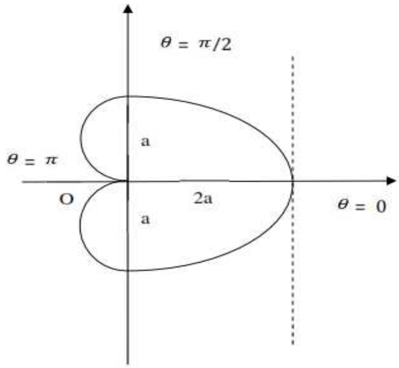
$\therefore\ \theta = \frac{\pi}{2} + \frac{\theta}{2}$. Hence when $\theta = 0, \phi = \frac{\pi}{2}$ and $r = 2a$.

Thus the tangent at $(2a, 0)$ is perpendicular to the initial line.

Since the maximum value of $r$ is $2a$, no portion of the curve lies to the right of the tangent at $(2a, 0)$ and hence the curve lies within the circle $r = 2a$.

The following table gives a set of points lying on the curve.

| $\theta$ | $0$ | $\pi/4$ | $\pi/2$ | $\pi$ | $-\pi/2$ | $-\pi/4$ |
|---|---|---|---|---|---|---|
| $r$ | $2a$ | $a\left(1 + \frac{1}{\sqrt{2}}\right)$ | $a$ | $0$ | $a$ | $a\left(1 + \frac{1}{\sqrt{2}}\right)$ |




When $\theta$ increases from 0 to $2\pi, r$ is positive and it decreases from $2a$ to 0.

The form of the curve is as shown in the figure and it is a cardioid.

**3.6 Taylor's series expansion**

Suppose that a function $f(x)$ has derivatives of all orders in some neighbourhood of the point $a$. Then $f(x)$ has the following Taylor's development with Lagrange's form of remainder $h$.

$f(a+h) = f(a) + h\,f'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h)$, where $0 < \theta < 1$. The term $R_n = \frac{h^n}{n!}f^{(n)}(a+\theta h)$, is called the Remainder after $n$ terms (Lagrange's form). If $\lim_{n\to\infty} R_n = 0$, then $f(x)$ has Taylor's series expansion at $a$ and is given by

$$f(a+h) = f(a) + h\,f'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \cdots$$

If we put $a + h = x$, so that $h = x - a$, then the Taylor's series expansion of $f(x)$ at $a$ takes the form

$$f(x) = f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots \quad (1)$$

If we put $a = 0$ in (1), we get

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots \quad (2)$$

and (2) is called the Maclaurin's series expansion of $f(x)$.

**Note** The Taylor's series converges to $f(x)$ if $\lim_{n\to\infty} R_n = 0$. We now give an example of a function $f$ whose Taylor's series does not converge to $f$.

Consider the function $f$ defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$$

Clearly $f$ has derivatives of all orders at every point $x \neq 0$ and $f^{(n)}(x) = P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}, x \neq 0$ where $P_n\left(\frac{1}{x}\right)$ is a polynomial in $\frac{1}{x}$.

We shall now show that for all $n, f^{(n)}(0) = 0$.

$$f'(0) \ = \lim_{h\to 0} \frac{f(h) - f(0)}{h}.$$

$$= \lim_{h\to 0} \frac{1}{h}\left(e^{-\frac{1}{h^2}}\right) = 0.$$

We assume that $f^{(n-1)}(0) = 0$ and prove $f^{(n)}(0) = 0$.

$$f^{(n)}(0) = \lim_{h\to 0} \frac{f^{(n-1)}(h) - f^{(n-1)}(0)}{h}.$$

$$= \lim_{h\to 0} \frac{1}{h} P_{n-1}\left(\frac{1}{h}\right) e^{-\frac{1}{h^2}} = 0.$$

Now, Taylor's series of $f$ around 0 is given by

$$f(0) + x\,f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \cdots$$

This series has sum zero (by [1]).

$\therefore$ The sum of the Taylor's series of $f(x)$ at any point $x \neq 0$ is different from $f(x)$.

**Taylor's series expansion of some standard functions**

**Result 1**

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

**Proof**

Let $f(x) = e^x$. Then $f'(x) = e^x$.

Hence $f(0) = 1$ and $f'(0) = 1$.

In general, $f^{(n)}(0) = 1$ for each $n$.

$$\therefore\ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

The above formula is valid for all $x \in \boldsymbol{R}$.

**Result 2**

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

**Proof**

Let $f(x) = \sin x$ . Then $f'(x) = \cos x, f''(x) = -\sin x, f^{(3)}(x) = -\cos x, \ldots \ldots \ldots$

In general $f^{(n)}(x) = \begin{cases} (-1)^{\frac{n}{2}} \sin x & if\ n\ is\ even \\ (-1)^{\frac{(n-1)}{2}} \cos x & if\ n\ is\ odd \end{cases}$

$\therefore f^{(n)}(0) = \begin{cases} 0 & if\ n\ is\ even \\ (-1)^{\frac{(n-1)}{2}} & if\ n\ is\ odd \end{cases}$

$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

The above formula is valid for all $x \in \boldsymbol{R}$.

**Result 3**

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ where $-1 < x < 1$.

**Proof**

Let $f(x) = \log(1+x)$ $\therefore f(0) = 0.$

$f'(x) = \frac{1}{1+x}$ $\therefore f'(0) = 1.$

$f''(x) = -\frac{1}{(1+x)^2}$ $\therefore f''(0) = -1.$

$f^{(3)}(x) = \frac{(-1)^2\, 2!}{(1+x)^3}$ $\therefore f^{(3)}(0) = (-1)^2\, 2!.$

……………….. ………………………..

$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ $\therefore f^{(n)}(0) = (-1)^{n-1}(n-1)!.$

$\therefore \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

This formula is valid for $|x| < 1$.

**Example 30**

Using Maclaurin's theorem expand $e^x \sec x$ as a power of $x$ upto the term containing $x^3$.

**Solution**

Let $f(x) = e^x \sec x$.

$\therefore f(0) = 1.$

(1)

$f'(x) = e^x \sec x\,(1 + \tan x).$

$\therefore f'(0) = 1.$

(2)

$f'(x) \;\; = e^x[\sec^3 x + (1 + \tan x)^2 \sec x].$

$= e^x \sec x\,(2\sec^2 x + 2\tan x).$

$\therefore f''(0) = 2.$

(3)

$f^{(3)}(x) = 2\,e^x(3\sec^3 x \tan x + 2\sec^3 x + \sec x \tan x + \tan^2 x \sec x.$

$f^{(3)}(0) = 4.$

(4)

By Maclaurin's Theorem,

$$e^x \sec x = 1 + x + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(4) + \cdots$$

$$= 1 + x + x^2 + \frac{2}{3}x^3 + \cdots$$

**Example 31**

Expand $\log \sin x$ in powers of $(x - 3)$.

**Solution**

Let $f(x) = \log \sin x$

Now $f(x) = \log \sin x$ $\qquad f(3) = \log \sin 3.$

$f'(x) = \frac{\cos x}{\sin x} = \cot x$ $\qquad f'(3) = \cot 3.$

$f''(x) = -\operatorname{cosec}^2 x$ $\qquad f''(3) = -\operatorname{cosec}^2 3.$

$f'''(x) = -2\operatorname{cosec} x\,(-\operatorname{cosec} x \cot x)$

$= 2\operatorname{cosec}^2 x \cot x$ $\qquad f'''(3) = 2\operatorname{cosec}^2 3 \cot 3.$

$$\therefore \log \sin x = \log \sin 3 + (x-3)\cot 3 - \frac{(x-3)^2}{2}\operatorname{cosec}^2 3 + \frac{(x-3)^3}{3}\operatorname{cosec}^2 3 \cot 3 + \cdots$$

**Example 32**

Expand $e^x$ in ascending powers of $(x - 1)$.

**Solution**

Let $f(x) = e^x$ $\qquad f(1) = e$

$f'(x) = e^x$ $\qquad f'(1) = e$

$f''(x) = e^x$ $\qquad f''(1) = e$

……………… $\qquad$ …………….

$\therefore e^x = e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots\right].$

**Example 33**

Use Taylor's Theorem to express the polynomial $2x^3 + 7x^2 + x - 6$ in powers of $(x-2)$.

**Solution**

Let $f(x) = 2x^3 + 7x^2 + x - 6$

Now $f(x) = 2x^3 + 7x^2 + x - 6$ $\qquad f(2) = 16 + 28 + 2 - 6 = 40$

$f'(x) = 6x^2 + 14x + 1$ $\qquad f'(2) = 24 + 28 + 1 = 53$

$f''(x) = 12x + 14$ $\qquad f''(2) = 24 + 14 = 38$

$f'''(x) = 12$ $\qquad f'''(2) = 12$

$\therefore 2x^3 + 7x^2 + x - 6 = 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3.$

**Example 34**

Find the expansion of $\log\sqrt{\frac{1+x}{1-x}}$.

**Solution**

$$\log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}[\log(1+x) - \log(1-x)].$$

$$= \frac{1}{2}\left[\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right)\right].$$

$$= \frac{1}{2}\left[2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)\right].$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

**Exercise 7**

Prove the following

1. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ for all $x \in \boldsymbol{R}$.

2. $\log(a + x) = \log a + \frac{x}{a} - \frac{1}{2}\frac{x^2}{a^2} + \cdots$ if $|x| < a$.

3. $\sin^2 x = x^2 - \frac{2^3 x^4}{4!} + \frac{2^5 x^6}{6!} - \cdots$ for all $x \in \boldsymbol{R}$.

4. Prove that $2 + x^2 - 3x^5 + 7x^6 = 7 + 29(x-1) + 76(x-1)^2 + 110(x-1)^3 + 90(x-1)^4 + 39(x-1)^5 + 7(x-1)^6$.

## UNIT-IV

## EVALUATION OF INTEGRALS

### 4.0 Introduction

Let $f: \boldsymbol{R} \to \boldsymbol{R}$ be a continuous function. A function $F: \boldsymbol{R} \to \boldsymbol{R}$ is called a primitive of $f(x)$ if $F'(x) = f(x)$ for all $x \in \boldsymbol{R}$.

We note that if $F(x)$ is a primitive for $f(x)$ then $F(x) + c$ where any arbitrary constant is also a primitive for $f(x)$.

Now, let $F'(x) = f(x)$ and let $f(x)$ be continuous. Then by theorems $\int_a^x f(x)dx = F(x) - F(a)$

(1)

and $\frac{d}{dx}\left(\int_a^x f(x)dx\right) = F'(x) = f(x)$

(2)

Equations (1) and (2) show that the process of integration and differentiation are inverse to each other for functions with continuous derivative. Also from (1), we see that the evaluation of the integral $\int_a^x f(x)dx$ requires acknowledge of a primitive of $f(x)$.

If $F(x)$ is a primitive of $f(x)$ we write $\int f(x)dx = F(x)$ and $\int f(x)dx$ is called an indefinite integral.

Thus evaluation of this indefinite integral is just determining a primitive of $f(x)$ when it exists.

In this chapter we develop various methods of evaluating indefinite integrals of various types of functions.

### 4.1 Some simple integrals

Consider $\int x^2\, dx$.

We note that $\frac{d}{dx}\left(\frac{x^3}{3} + c\right) = x^2$ where $c$ is any arbitrary constant.

$\therefore \frac{x^3}{3} + c$ is a primitive of $x^2$.

$\therefore \int x^2\, dx = \frac{x^3}{3} + c.$

**Note** Hereafter in evaluating indefinite integrals we shall take the constant $c$ to be understood after the primitive.

We give below a list of standard integrals which are immediate consequence of the corresponding formulae for differentiation.

1. $\int k\, dx = kx.$

2. $\int x^n \, dx = \frac{x^{n+1}}{n+1}$ when $n \neq -1$.

3. $\int \left(\frac{1}{x}\right) \, dx = \log x$.

4. $\int e^x \, dx = e^x$.

5. $\int a^x \, dx = \frac{a^x}{\log a}$.

6. $\int \sin x \, dx = -\cos x$.

7. $\int \cos x \, dx = \sin x$.

8. $\int \sec^2 x \, dx = \tan x$.

9. $\int \operatorname{cosec}^2 x \, dx = -\cot x$.

10. $\int \sec x \tan x \, dx = \sec x$.

11. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x$.

12. $\int \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1} x$.

13. $\int \frac{dx}{1+x^2} = \tan^{-1} x$.

14. $\int \frac{dx}{x\sqrt{(x^2-1)}} = \sec^{-1} x$.

15. $\int \sinh x \, dx = \cosh x$.

16. $\int \cosh x \, dx = \sinh x$.

17. $\int \frac{dx}{\sqrt{(1+x^2)}} = \sinh^{-1} x = \log\left[x + \sqrt{(x^2+1)}\right]$.

18. $\int \frac{dx}{\sqrt{(x^2-1)}} = \cosh^{-1} x = \log\left[x + \sqrt{(x^2-1)}\right]$.

**Note** (i) $\int a\, f(x)dx = a \int f(x)dx$ where $a \in \boldsymbol{R}$.

(ii) $\int [f(x) \pm g(x)] \, dx = \int f(x)dx \pm \int g(x)dx$.

**Example 1**

Evaluate $I = \int \left[e^x + 8\cos x - \frac{4}{\sqrt{(1-x^2)}} + 8x^2\right] dx$.

**Solution**

$I = \int e^x \, dx + 8\int \cos x \, dx - 4\int \frac{dx}{\sqrt{(1-x^2)}} + 8\int x^2 \, dx$.

$= e^x + 8\sin x - 4\sin^{-1} x + \frac{8}{3}x^3$.

**Example 2**

Evaluate $= \int \frac{dx}{1+\sin x}$.

**Solution**

$I = \int \left(\frac{1-\sin x}{\cos^2 x}\right) dx.$

$= \int (\sec^2 x - \sec x \tan x) dx.$

$= \tan x - \sec x.$

**Exercise 1**

Evaluate the following integrals

1. $ax^5 + b\sqrt{x} + c\sin x + k\,e^x.$ 2. $\frac{(x^2+4x)(3-4x)}{x^3}$

3. $\frac{1}{1-\cos x}$ 4. $\frac{\cos^2 x}{1-\sin x}$ 5. $e^x - \frac{3}{x} + \frac{4}{1+x^2}.$

**Answers**

1. $\frac{1}{6}ax^6 + \frac{2b}{3}x^{3/2} - c\cos x + ke^x$ 2. $-13\log x - 4x - 12x^{-1}$

3. $-(\cot x + \operatorname{cosec} x)$ 4. $x - \cos x$

5. $e^x - \log x^3 + 4\tan^{-1} x$

**4.2 Method of Substitution**

A standard method of evaluating a given integral is to reduce it to a standard formula listed in 4.1 by a simple substitution. In many cases the form of the function helps us to find or guess a suitable substitution. However in general, finding a suitable substitution for evaluating an integral need experience and practice.

**Some standard integrals**

**1.** $\int \frac{dx}{\sqrt{(a^2-x^2)}} = \sin^{-1}(x/a)$

**Proof**

Put $x = a\sin\theta$. Hence $dx = a\cos\theta\, d\theta$.

$\therefore \int \frac{dx}{\sqrt{(a^2-x^2)}} = \int \frac{a\cos\theta\, d\theta}{a\cos\theta} = \int d\theta = \theta.$

$= \sin^{-1}(x/a).$

**2.** $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}(x/a)$

**Proof**

Put $x = a\tan\theta$. Hence $dx = a\sec^2\theta\, d\theta$.

$\therefore \int \frac{dx}{a^2+x^2} = \int \frac{a\sec^2\theta\, d\theta}{a^2\sec^2\theta} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta.$

$= \frac{1}{a}\tan^{-1}(x/a).$

**3.** $\int \frac{dx}{\sqrt{(a^2+x^2)}} = \sinh^{-1}(x/a)$

**Proof**

Put $x = a\sinh\theta$. Hence $dx = a\cosh\theta\, d\theta$.

$\therefore \int \frac{dx}{\sqrt{(a^2+x^2)}} = \int \frac{a\cosh\theta\, d\theta}{a\cosh\theta} = \int d\theta = \theta.$

$= \sinh^{-1}(x/a).$

**4.** $\int \frac{dx}{\sqrt{(x^2-a^2)}} = \cosh^{-1}(x/a)$

**Proof**

Put $x = a\cosh\theta$. Hence $dx = a\sinh\theta\, d\theta$.

$\therefore \int \frac{dx}{\sqrt{(x^2-a^2)}} = \int \frac{a\sinh\theta\, d\theta}{a\sinh\theta} = \int d\theta = \theta.$

$= \cosh^{-1}(x/a).$

**5.** $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$

**Proof**

Let $\frac{1}{x^2-a^2} = \frac{A}{x-a} + \frac{B}{x+a}$.

Then $A = \frac{1}{2a}$ and $= -\frac{1}{2a}$.

$\therefore \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int \frac{dx}{x-a} - \frac{1}{2a}\int \frac{dx}{x+a}$

$= \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right).$

**6.** $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$

**Proof**

Let $\frac{1}{a^2-x^2} = \frac{A}{a-x} + \frac{B}{a+x}$.

Then $A = \frac{1}{2a}$ and $= \frac{1}{2a}$.

$\therefore \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\int \frac{dx}{a-x} + \frac{1}{2a}\int \frac{dx}{a+x}$

$= \frac{1}{2a}\log\left(\frac{a-x}{a+x}\right).$

**7.** $\int \sqrt{(a^2-x^2)}\, dx = \frac{1}{2}x\sqrt{(a^2-x^2)} + \frac{1}{2}a^2\sin^{-1}(x/a)$

**Proof**

Put $x = a\sin\theta$. Hence $dx = a\cos\theta\ d\theta$.

$$\therefore \int\sqrt{(a^2 - x^2)}\,dx = a^2\int\cos^2\theta\ d\theta.$$

$$= \frac{1}{2}a^2\int(1+\cos 2\theta)d\theta.$$

$$= \frac{1}{2}a^2\left[\theta + \frac{1}{2}\sin 2\theta\right].$$

$$= \frac{1}{2}a^2[\theta + \sin\theta\cos\theta]$$

$$= \frac{1}{2}a^2\sin^{-1}(x/a) + \frac{1}{2}a^2(x/a)\sqrt{[1-(x/a)^2]}$$

$$= \frac{1}{2}x\sqrt{(a^2 - x^2)} + \frac{1}{2}a^2\sin^{-1}(x/a).$$

**8.** $\int\sqrt{(a^2 + x^2)}\,dx = \frac{1}{2}x\sqrt{(a^2 + x^2)} + \frac{1}{2}a^2\sinh^{-1}(x/a)$

**Proof**

Put $x = a\sinh\theta$. Hence $dx = a\cosh\theta\ d\theta$.

$$\therefore \int\sqrt{(a^2 + x^2)}\,dx = a^2\int\cosh^2\theta\ d\theta.$$

$$= \frac{1}{2}a^2\int(1+\cosh 2\theta)d\theta.$$

$$= \frac{1}{2}a^2\left[\theta + \frac{1}{2}\sinh 2\theta\right].$$

$$= \frac{1}{2}a^2[\theta + \sinh\theta\cosh\theta]$$

$$= \frac{1}{2}x\sqrt{(a^2 + x^2)} + \frac{1}{2}a^2\sinh^{-1}(x/a).$$

**9.** $\int\sqrt{(x^2 - a^2)}\,dx = \frac{1}{2}x\sqrt{(x^2 - a^2)} - \frac{1}{2}a^2\cosh^{-1}(x/a)$

**Proof**

Put $x = a\cosh\theta$. Hence $dx = a\sinh\theta\ d\theta$.

$$\therefore \int\sqrt{(x^2 - a^2)}\,dx = a^2\int\sinh^2\theta\ d\theta.$$

$$= \frac{1}{2}a^2\int(\cosh 2\theta - 1)d\theta.$$

$$= \frac{1}{2}a^2\left[\frac{1}{2}\sinh 2\theta - \theta\right].$$

$$= \frac{1}{2}x\sqrt{(x^2 - a^2)} - \frac{1}{2}a^2\cosh^{-1}(x/a).$$

**10.** $\int\tan x\ dx = \log(\sec x)$

**Proof**

$$\int\tan x\ dx = \int\frac{\sin x}{\cos x}\ dx$$

$$= -\int \frac{d(\cos x)}{\cos x} = -\log(\cos x)$$
$$= \log(\sec x).$$

**11.** $\int \boldsymbol{\cot x \; dx = \log(\sin x)}$

Proof is similar to the previous problem.

**12.** $\int \boldsymbol{\sec x \; dx = \log(\sec x + \tan x)}$

**Proof**

$$\int \sec x \; dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$$
$$= \int \frac{d(\sec x + \tan x)}{\sec x + \tan x}$$
$$= \log(\sec x + \tan x).$$

**13.** $\int \boldsymbol{\mathrm{cosec}\, x \; dx = -\log(\mathrm{cosec}\, x + \cot x)}$

Proof is similar to the previous problem.

**Example 3**

Evaluate $\int \frac{\sin\sqrt{x}}{\sqrt{x}} \; dx$

**Solution**

Put $y = \sqrt{x}$

$dy = \frac{dx}{2\sqrt{x}}$

Hence $\int \frac{\sin\sqrt{x}}{\sqrt{x}} \; dx = 2\int \sin y \; dy$

$$= -2\cos y = -2\cos\sqrt{x}.$$

**Example 4**

Evaluate $\int x \, e^{x^2} \; dx.$

**Solution**

$$\int x \, e^{x^2} \; dx = \frac{1}{2}\int e^{x^2} \; d(x^2).$$
$$= \frac{1}{2} e^{x^2}.$$

**Example 5**

Evaluate $\int \frac{x \, dx}{1+x^4}$

**Solution**

Put $y = x^2$. Hence $dy = 2x\,dx$.

$$\therefore \int \frac{x\,dx}{1+x^4} = \frac{1}{2}\int \frac{dy}{1+y^2}$$
$$= \frac{1}{2}\tan^{-1} y$$
$$= \frac{1}{2}\tan^{-1}(x^2).$$

**Example 6**

Evaluate $\int a^{x^2}\, x\, dx$.

**Solution**

$$\int a^{x^2}\, x\, dx = \frac{1}{2}\int a^t\, dt \text{ (putting } x^2 = t)$$
$$= \frac{1}{2}\int e^{t\log a} dt = \frac{e^{t\log a}}{2\log a}$$
$$= \frac{a^{x^2}}{2\log a}.$$

**Example 7**

Prove that $\int_0^1 \frac{dx}{e^x+1} = \log\left(\frac{2e}{1+e}\right)$

**Solution**

$$\int_0^1 \frac{dx}{e^x+1} = \int_0^1 \frac{e^{-x}}{1+e^{-x}}\, dx$$
$$= -\int_0^1 \frac{d(e^{-x})}{1+e^{-x}}\, dx$$
$$= -[\log(1+e^{-x})]_0^1 = \log 2 - \log(1+1/e)$$
$$= \log\left(\frac{2e}{1+e}\right).$$

**Exercise 2**

Integrate the following functions with respect to $x$.

1. $x^2\cos(x^3)$
2. $\frac{1}{x\log x}$
3. $\frac{(\log x)^3}{x}$
4. $\frac{x}{1+x^2}$
5. $\frac{x}{(5x^2-3)^{-7}}$
6. $\frac{\sin^{-1} x}{\sqrt{(1-x^2)}}$
7. $\frac{\cos x}{\sqrt{(1+\sin^2 x)}}$
8. $\frac{x+1}{x^2+2x+3}$
9. $\frac{e^x}{\sqrt{(1-e^{2x})}}$
10. $\frac{1}{\sqrt{(9+25\,x^2)}}$
11. $\frac{1}{49x^2+16}$
12. $\sqrt{25+x^2}$

**Answers**

1. $\frac{1}{3}\sin(x^3)$ 2. $\log(\log x)$ 3. $\frac{1}{4}(\log x)^4$

4. $\frac{1}{2}\log(1+x^2)$ 5. $\frac{1}{80}(5x^2-3)^8$ 6. $\frac{1}{2}(\sin^{-1}x)^2$

7. $\sinh^{-1}(\sin x)$ or $\log\left[\sin x+\sqrt{1+\sin^2 x}\right]$

8. $\frac{1}{2}\log(x^2+2x+3)$ 9. $\sin^{-1}(e^x)$

10. $\frac{1}{5}\sinh^{-1}(5x/3)$ 11. $\frac{1}{28}\tan^{-1}(7x/4)$

12. $\frac{1}{2}x\sqrt{25+x^2}+\frac{25}{2}\sinh^{-1}(x/5)$.

## 4.3 Integration of Rational functions

In this section we discuss various methods of evaluation $\int \boldsymbol{R}(x)dx$ where $\boldsymbol{R}(x)$ denotes the ratio of two polynomials in $x$.

**Type 1** $\int \frac{dx}{ax^2+bx+c}$

We note that $ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$

$$= a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2+\lambda^2\right] \qquad \text{where } \lambda^2 = \pm\frac{4ac-b^2}{4a^2}$$

∴ The given integral reduces to $\frac{1}{a}\int\frac{du}{u^2\pm\lambda^2}$ where $u = x+\frac{b}{2a}$ and this integral can be easily evaluated.

**Type 2** $\int \frac{lx+m}{ax^2+bx+c}\,dx$

Let $lx+m = A(2ax+b)+B$

Equating like terms we get $A=\frac{1}{2a}$ and $B=\left(m-\frac{lb}{2a}\right)$

∴ The given integral reduces to $\frac{l}{2a}\int\frac{d(ax^2+bx+c)}{ax^2+bx+c}+\left(m-\frac{lb}{2a}\right)\int\frac{dx}{ax^2+bx+c}$ which can be evaluated by using type 1.

**Note**

1. A rational function $\boldsymbol{R}(x)$ is called a proper rational function if the degree of the numerator is smaller than the degree of the denominator. In some cases a proper rational function $\boldsymbol{R}(x)$ can be resolved into partial fractions and $\int \boldsymbol{R}(x)dx$ can be evaluated using types 1 or 2.

2. If $\boldsymbol{R}(x)$ is not a proper rational function then $\boldsymbol{R}(x)$ can be expressed as a sum of a polynomial and a proper rational function by ordinary division.

We illustrate these methods in the following problems.

**Example 8**

Evaluate $\int \frac{dx}{x^2-6x+5}$

**Solution**

$$\int \frac{dx}{x^2-6x+5} = \int \frac{dx}{(x-3)^2-2^2}$$
$$= \frac{1}{4}\log\left(\frac{x-3-2}{x-3+2}\right)$$
$$= \frac{1}{4}\log\left(\frac{x-5}{x-1}\right).$$

**Example 9**

Evaluate $\int \frac{dx}{3x^2-2x+2}$

**Solution**

$$\int \frac{dx}{3x^2-2x+2} = \frac{1}{3}\int \frac{dx}{x^2-\frac{2}{3}x+\frac{2}{3}}$$
$$= \frac{1}{3}\int \frac{dx}{\left(x-\frac{1}{3}\right)^2+\frac{2}{3}-\frac{1}{9}}$$
$$= \frac{1}{3}\int \frac{dx}{\left(x-\frac{1}{3}\right)^2+\left(\sqrt{5/3}\right)^2}$$
$$= \frac{3}{3\sqrt{5}}\tan^{-1}\left(\frac{x-\frac{1}{3}}{\sqrt{5}/3}\right)$$
$$= \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{3x-1}{\sqrt{5}}\right).$$

**Example 10**

Evaluate $\int \frac{2x-1}{5x^2-x+2}\,dx$

**Solution**

Let $2x-1 = A(10x-1)+B$

Equating the coefficients of x and constant terms we get $A = \frac{1}{5}$ and $B = -\frac{4}{5}$

$$\therefore \int \frac{2x-1}{5x^2-x+2}\,dx = \frac{1}{5}\int \frac{10x-1}{5x^2-x+2}\,dx - \frac{4}{5}\int \frac{dx}{5x^2-x+2}$$

$$= \frac{1}{5}\log(5x^2 - x + 2) - \frac{4}{25}\int \frac{dx}{x^2 - \frac{1}{5}x + \frac{2}{5}}$$
$$= \frac{1}{5}\log(5x^2 - x + 2) - \frac{4}{25}\int \frac{dx}{\left(x - \frac{1}{10}\right)^2 + \left(\sqrt{39/10}\right)^2}$$
$$= \frac{1}{5}\log(5x^2 - x + 2) - \frac{4\times 10}{25\times\sqrt{39}}\tan^{-1}\left(\frac{x - \frac{1}{10}}{\sqrt{(39/10)}}\right)$$
$$= \frac{1}{5}\log(5x^2 - x + 2) - \frac{8}{5\sqrt{39}}\tan^{-1}\left(\frac{10x - 1}{\sqrt{39}}\right)$$

**Example 11**

Evaluate $I = \int \frac{x}{(x-a)(x-b)}\, dx$

**Solution**

Let $\frac{x}{(x-a)(x-b)}\, dx = \frac{A}{x-a} + \frac{B}{x-b}$.

We get $A = \frac{a}{a-b}$ and $B = \frac{b}{b-a}$

$\therefore I = \frac{a}{a-b}\int \frac{dx}{x-a} - \frac{b}{b-a}\int \frac{dx}{x-b}$

$= \frac{1}{a-b}[a\log(x-a) - b\log(x-b)]$

$= \frac{1}{a-b}\log\left[\frac{(x-a)^a}{(x-b)^b}\right]$.

**Example 12**

Evaluate $\int \frac{x^{27}}{x^{14}+4}\, dx$

**Solution**

$\frac{x^{27}}{x^{14}+4} = x^{13} - \frac{4x^{13}}{x^{14}+4}$

$\therefore \int \frac{x^{27}}{x^{14}+4}\, dx = \int x^{13}\, dx - 4\int \frac{x^{13}}{x^{14}+4}\, dx$

$= \frac{x^{14}}{14} - \frac{4}{14}\log(x^{14} + 4)$

$= \frac{x^{14}}{14} - \frac{2}{7}\log(x^{14} + 4)$.

**Exercise 3**

Integrate the following functions with respect to $x$.

1. $\frac{1}{x^2+4x+10}$ 2. $\frac{2x+1}{x^2+21x+3}$ 3. $\frac{1}{(x+1)(x+2)}$

4. $\frac{x-8}{x^3-4x^2+4x}$ 5. $\frac{2}{(1+x^2)(1-x)}$ 6. $\frac{x}{x^3-1}$

7. $\frac{x^2+1}{(x^2-1)(2x+1)}$ 8. $\frac{x^5}{x^3-1}$

9. show that $\int_0^1 \frac{dx}{1+x^3} = \frac{1}{3}\log 2 + \frac{\pi}{3\sqrt{3}}$.

**Answers**

1. $\frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{x+2}{\sqrt{6}}\right)$ 2. $\log(x^2+21x+3) - \frac{20}{\sqrt{429}}\log\left(\frac{2x+21-\sqrt{429}}{2x+21+\sqrt{429}}\right)$

3. $\log\left(\frac{x+1}{x+2}\right)$ 4. $\frac{3}{x-2} + \log\frac{(x-2)^2}{x^2}$

5. $-\log(1-x) + \frac{1}{2}\log(x^2+1) + \tan^{-1}x$

6. $\frac{1}{6}\log(x^2-x+1) - \frac{1}{3}\log(x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$

7. $\frac{1}{3}\log(x-1) + \log(x+1) - \frac{5}{6}\log(2x+1)$

8. $\frac{1}{3}[x^3 + \log(x^3-1)]$

## 4.4 Integration of Irrational functions

**Type 1** $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

As in type 1 of 4.3 the integral can be reduced to one of the forms $\int \frac{du}{\sqrt{u^2+\lambda^2}}$ or $\int \frac{du}{\sqrt{u^2-\lambda^2}}$ or $\int \frac{du}{\sqrt{\lambda^2-u^2}}$ which are simple integrals.

**Type 2** $\int \frac{lx+m}{\sqrt{ax^2+bx+c}}\, dx$

As in type 2 of 4.3 the given integral becomes $\frac{l}{2a}\int \frac{d(ax^2+bx+c)}{ax^2+bx+c} + \left(m - \frac{lb}{2a}\right)\int \frac{dx}{ax^2+bx+c}$ which can be evaluated using type 1.

**Type 3** $\int \sqrt{ax^2+bx+c}\, dx$

This integral can be reduced to one of the forms $\int \sqrt{u^2-\lambda^2}\, du$ or $\int \sqrt{\lambda^2-u^2}\, du$ or $\int \sqrt{u^2+\lambda^2}\, du$ which can be evaluated by a suitable substitution.

Type 4 $\int (lx+m)\sqrt{ax^2+bx+c}\, dx$

Let $lx+m = A(2ax+b)+B$.

$\therefore A = \frac{1}{2a}$ and $B = \left(m - \frac{lb}{2a}\right)$.

Hence the given integral reduces to $\frac{1}{2a}\int \sqrt{ax^2+bx+c}\, d(ax^2+bx+c) + \left(m - \frac{lb}{2a}\right)\int \sqrt{ax^2+bx+c}\, dx$ which can be evaluated.

**Type 5** $\int \frac{\boldsymbol{dx}}{(\boldsymbol{x-k})\sqrt{\boldsymbol{ax^2+bx+c}}}$

This can be reduced to the form $\int \frac{dx}{\sqrt{Ax^2+Bx+C}}$ by the substitution $-k = \frac{1}{t}$.

**Type 6** $\int \frac{\boldsymbol{dx}}{(\boldsymbol{px^2+q})\sqrt{\boldsymbol{ax^2+bx+c}}}$

This can be reduced to one of the types discussed earlier by the substitution $= \frac{1}{t}$.

**Type 7** $\int \sqrt{(\boldsymbol{x-a})(\boldsymbol{b-x})}\,\boldsymbol{dx}$; $\int \frac{\boldsymbol{dx}}{\sqrt{(\boldsymbol{x-a})(\boldsymbol{b-x})}}$; $\int \sqrt{\left(\frac{\boldsymbol{x-a}}{\boldsymbol{b-x}}\right)}\,\boldsymbol{dx}$ can be evaluated by using the substitution $x = a\sin^2\theta + b\cos^2\theta$.

We illustrate theses methods in the following problems.

**Example 13**

Evaluate $\int \frac{dx}{\sqrt{2+3x-2x^2}}$.

**Solution**

$$\int \frac{dx}{\sqrt{2+3x-2x^2}} = \int \frac{dx}{\sqrt{-2\left(x^2-\frac{3}{2}x-1\right)}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{-\left[\left(x-\frac{3}{4}\right)^2-\left(\frac{5}{4}\right)^2\right]}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left[\left(\frac{5}{4}\right)^2-\left(x-\frac{3}{4}\right)^2\right]}}$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x-\frac{3}{4}}{5/4}\right)$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4x-3}{5}\right).$$

**Example 14**

Evaluate $\int \sqrt{\left(\frac{3-2x}{1-x}\right)}\,dx$

**Solution**

$$\int \sqrt{\left(\frac{3-2x}{1-x}\right)}\,dx = \int \frac{3-2x}{\sqrt{(1-x)(3-2x)}}\,dx$$

$$= \int \frac{3-2x}{\sqrt{(3-5x-2x^2)}}\,dx$$

Let $3-2x = A(4x-5)+B$

Equating the like terms we get $A = -\frac{1}{2}$ and $B = \frac{1}{2}$

$$\therefore \int \frac{3-2x}{\sqrt{2x^2-5x+3}}\,dx = -\frac{1}{2}\int \frac{(4x-5)}{\sqrt{2x^2-5x+3}}\,dx + \frac{1}{2}\int \frac{dx}{\sqrt{2x^2-5x+3}}$$

$$= -\sqrt{2x^2-5x+3} + \frac{1}{2\sqrt{2}}\int \frac{dx}{\sqrt{\left[\left(x-\frac{5}{4}\right)^2-\left(\frac{1}{4}\right)^2\right]}}$$

$$= -\sqrt{2x^2-5x+3} + \frac{1}{2\sqrt{2}}\cosh^{-1}(4x-5).$$

**Example 15**

Evaluate $\int_0^2 \sqrt{\left(\frac{2+x}{2-x}\right)}\,dx$

**Solution**

$$I = \int_0^2 \sqrt{\left(\frac{2+x}{2-x}\right)}\,dx = \int_0^2 \frac{2+x}{\sqrt{(2+x)(2-x)}}dx$$

$$= \int_0^2 \frac{2+x}{\sqrt{4-x^2}}dx$$

$$= \int_0^2 \frac{2\,dx}{\sqrt{4-x^2}} + \int_0^2 \frac{x\,dx}{\sqrt{4-x^2}}$$

$$= \left[2\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 - \left[\sqrt{(4-x^2)}\right]_0^2$$

$$= \pi + 2.$$

**Example 16**

Evaluate $\int (3x+2)\sqrt{x^2+x+1}\,dx$

**Solution**

Let $(3x+2) = A\frac{d}{dx}(x^2+x+1)+B$

$= A(2x+1)+B.$

Comparing like terms we get $A = \frac{3}{2}$ and $= \frac{1}{2}$.

$$\int (3x+2)\sqrt{x^2+x+1}\,dx = \int \left[\frac{3}{2}(2x+1)+\frac{1}{2}\right]\sqrt{x^2+x+1}\,dx$$

$$= \frac{3}{2}\int (2x+1)\sqrt{x^2+x+1}\,dx + \frac{1}{2}\int \sqrt{x^2+x+1}\,dx$$

$$= \frac{3}{2}\frac{(x^2+x+1)^{3/2}}{3/2} + \frac{1}{2}\int \sqrt{\left[\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right]}\,dx$$

$$= (x^2+x+1)^{3/2} + \frac{1}{4}\left[\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} + \frac{3}{4}\sinh^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]$$

$$= (x^2 + x + 1)^{3/2} + \frac{1}{4}\left[\left(\frac{2x+1}{2}\right)\sqrt{x^2 + x + 1} + \frac{3}{4}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right]$$

**Example 17**

Evaluate $I = \int \frac{dx}{(x^2-1)\sqrt{x^2+1}}$

**Solution**

Put $= \frac{1}{t}$. Hence $dx = -\frac{1}{t^2}\ dt.$

Now, $I = \int \frac{dx}{(x^2-1)\sqrt{x^2+1}} = -\int \frac{dt/t^2}{(1/t^2-1)\sqrt{(1/t^2+1)}}.$

$$= -\int \frac{t\ dt}{(1-t^2)\sqrt{1+t^2}}.$$

Now put $1 + t^2 = y^2$. Hence $t\ dt = y\ dy.$

Hence $I = -\int \frac{y\ dy}{(2-y^2)y}$

$$= \int \frac{dy}{y^2-(\sqrt{2})^2}$$

$$= \frac{1}{2\sqrt{2}}\log\left(\frac{y-\sqrt{2}}{y+\sqrt{2}}\right)$$

$$= \frac{1}{2\sqrt{2}}\log\left[\frac{\sqrt{1+t^2}-\sqrt{2}}{\sqrt{1+t^2}+\sqrt{2}}\right]$$

$$= \frac{1}{2\sqrt{2}}\log\left[\frac{\sqrt{(1+x^2)}-\sqrt{2}x}{\sqrt{(1+x^2)}+\sqrt{2}x}\right].$$

**Exercise 4**

Integrate the following with respect to $x$.

1. $\frac{1}{\sqrt{(5-7x-3x^2)}}$
2. $\frac{2ax+b}{\sqrt{(ax^2+bx+c)}}$
3. $\frac{x+3}{\sqrt{(3+4x-4x^2)}}$
4. $\sqrt{\left(\frac{x-1}{2x-3}\right)}$
5. $\sqrt{(3x^2 + 4x + 1)}$
6. $\frac{1}{(1+x^2)\sqrt{(x^2+x+1)}}$
7. $\frac{1}{(2x^2+3)\sqrt{(3x^2-4)}}$
8. $\frac{1}{x^2\sqrt{(4+x^2)}}$
9. Evaluate $\int_0^{1/2} \frac{(3x+7)dx}{\sqrt{1-x-x^2}}$

**Answers**

1. $\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{6x+7}{\sqrt{109}}\right)$
2. $2\sqrt{ax^2 + bx + c}$
3. $-\frac{1}{4}\sqrt{3 + 4x - 4x^2} + \frac{7}{4}\sin^{-1}\left(\frac{2x-1}{2}\right)$
4. $\frac{1}{2}\sqrt{2x^2 + x - 3} - \frac{5}{4\sqrt{2}}\cosh^{-1}\left(\frac{4x+1}{5}\right)$

5. $\left(\frac{3x+2}{6}\right)\sqrt{3x^2+4x+1}+\frac{\sqrt{3}}{18}\cosh^{-1}(3x+2)$

6. $-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right)$

7. $\frac{1}{2\sqrt{51}}\log\left[\frac{\sqrt{17}x+\sqrt{3}\sqrt{3x^2-4}}{\sqrt{17}x-\sqrt{3}\sqrt{3x^2-4}}\right]$

8. $-\frac{\sqrt{4+x^2}}{4x}$

9. $\frac{3}{2}+\frac{11}{2}\left[\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)-\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)\right]$

## 4.5 Integration of Trigonometric Functions

In general an integral of the form $\int \boldsymbol{R}(\sin x, \cos x)dx$ can be evaluated by putting $t=\tan(x/2)$ so that $\sin x=\frac{2t}{1+t^2}$; $\cos x=\frac{1-t^2}{1+t^2}$ and $=\frac{2\,dt}{1+t^2}$.

However there are some special methods if the functions involved are simple.

For example $\int \frac{a\sin x+b\cos x}{l\sin x+m\cos x}\,dx$ can be evaluated by putting $a\sin x+b\cos x=A\frac{d}{dx}(l\sin x+m\cos x)+B(l\sin x+m\cos x)$.

These methods are illustrated in the following problems.

**Example 18**

Evaluate $I=\int \frac{2\sin x+\cos x}{3\sin x+\cos x}\,dx$

**Solution**

Let $2\sin x+\cos x=A\frac{d}{dx}(3\sin x+\cos x)+B(3\sin x+\cos x)$

$=A(3\cos x-\sin x)+B(3\sin x+\cos x)$

Equating like terms we get $3A+B=1$ and $-A+3B=2$

Solving we get $A=\frac{1}{10}$ and $B=\frac{7}{10}$

$\therefore I=\frac{1}{10}\int \frac{3\cos x-\sin x}{3\sin x-\cos x}\,dx+\frac{7}{10}\int dx$

$=\frac{1}{10}\log(3\sin x+\cos x)+\frac{7}{10}x$

**Example 19**

Show that $\int_0^{\pi/2}\frac{dx}{3\cos x+4\sin x}=\frac{1}{5}\log 6$

**Solution**

Put $t=\tan(x/2)$

$dt=\frac{1}{2}\sec^2(x/2)\,dx$

$\therefore dx = \frac{2\,dt}{1+t^2}$

$$\int_0^{\pi/2} \frac{dx}{3\cos x + 4\sin x} = \int_0^1 \frac{2\,dt}{(1+t^2)\left[3\left(\frac{1-t^2}{1+t^2}\right)+4\left(\frac{2t}{1+t^2}\right)\right]}$$

$$= 2\int_0^1 \frac{dt}{3-3t^2+8t}$$

$$= \frac{2}{3}\int_0^1 \frac{dt}{\left(\frac{5}{3}\right)^2-\left(t-\frac{4}{3}\right)^2}$$

$$= \left[\left(\frac{2}{3}\right)\times\left(\frac{1}{2}\right)\times\left(\frac{3}{5}\right)\log\left(\frac{\frac{5}{3}+t-\frac{4}{3}}{\frac{5}{3}-t+\frac{4}{3}}\right)\right]_0^1$$

$$= \left[\frac{1}{5}\log\frac{t+\frac{1}{3}}{3-t}\right]_0^1$$

$$= \frac{1}{5}\left[\log\frac{2}{3} - \log\frac{1}{9}\right]$$

$$= \frac{1}{5}\log 6.$$

**Example 20**

Evaluate $I = \int \frac{dx}{1+a^2\cos^2 x+b^2\sin^2 x}$

**Solution**

$$I = \int \frac{dx}{1+a^2\cos^2 x+b^2\sin^2 x}.$$

$$= \int \frac{\sec^2 x dx}{\sec^2 x+a^2+b^2\tan^2 x}.$$

Put $t = \tan x$. Hence $dt = \sec^2 x\,dx$.

$$\therefore I = \int \frac{dx}{(1+t^2)+a^2+b^2t^2}$$

$$= \int \frac{dt}{(1+b^2)t^2+a^2+1}.$$

$$= \frac{1}{1+b^2}\int \frac{dt}{t^2+\left(\sqrt{\left(\frac{a^2+1}{b^2+1}\right)}\right)^2}.$$

$$= \frac{1}{1+b^2}\left[\sqrt{\left(\frac{b^2+1}{a^2+1}\right)}\tan^{-1}\left(\sqrt{\left(\frac{b^2+1}{a^2+1}\right)}\right)t\right].$$

**Exercise 5**

Integrate the following

1. $\frac{7\sin x}{5\ \cos x+2\sin x}$
2. $\frac{5\cos x}{2\cos x+\sin x}$
3. $\frac{1}{1-3\sin x}$
4. $\frac{\sin x}{5+4\sin x}$
5. $\frac{\cos x}{5-3\cos x}$
6. $\frac{1}{1+3\cos^2 x}$

7. $\frac{1}{3\cos^2 x+11\sin^2 x}$ 8. Show that $\int_{\pi/4}^{3\pi/4} \frac{dx}{2\cos^2 x+1} = \frac{2}{3\sqrt{3}}$

**Answers**

1. $\frac{14}{26}x - \frac{35}{26}\log(5\cos x + 2\sin x)$ 2. $2x + \log(2\cos x + \sin x)$

3. $\frac{1}{2\sqrt{2}}\log\left(\frac{\tan(x/2)-3-2\sqrt{2}}{\tan(x/2)-3+2\sqrt{2}}\right)$ 4. $\frac{1}{4}x - \frac{5}{6}\tan^{-1}\left[\frac{1}{3}\left(5\tan\left(\frac{x}{2}\right)+4\right)\right]$

5. $\frac{5}{6}\tan^{-1}\left[2\tan\left(\frac{x}{2}\right)\right] - \frac{1}{3}x$ 6. $\frac{1}{2}\tan^{-1}\left[\frac{1}{2}\tan x\right]$

7. $\frac{1}{2\sqrt{33}}\log\left(\frac{\sqrt{(3/11)}-\tan x}{\sqrt{(3/11)}+\tan x}\right)$

## 4.6 Evaluation of Definite Integrals

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

We now prove some more properties of definite integrals.

**3.** $\boldsymbol{\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & if\ f(x) is\ even \\ 0 & if\ f(x)\ is\ odd \end{cases}}$

**Proof**

$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$ (by 2)

$= -\int_a^0 f(-y)dy + \int_0^a f(x)dx$ (by putting $x = -y$)

$= \int_0^a f(-y)dy + \int_0^a f(x)dx$

(1)

**Case (i)** Let $f(x)$ be an even function

$\therefore f(x) = f(-x)$

From (1) we get $\int_{-a}^{a} f(x)dx = \int_0^a f(y)dy + \int_0^a f(x)dx$

$= 2\int_0^a f(x)dx.$

**Case (ii)** Let $f(x)$ be an odd function

$\therefore f(x) = -f(-x)$

From (1) we get $\int_{-a}^{a} f(x)dx = -\int_0^a f(y)dy + \int_0^a f(x)dx = 0.$

**4.** $\boldsymbol{\int_0^a f(x)dx = \int_0^a f(a-x)dx}$

**Proof**

Putting $a - x = y$ in the right hand side integral we get the required result.

**5.** $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$

**Proof**

$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$ (by 2)

$= \int_0^a f(x)dx - \int_a^0 f(2a-y)dy$ (putting $y = 2a - x$)

$= \int_0^a f(x)dx + \int_a^0 f(2a-x)dx$ (1)

**Case (i)** Let $f(2a-x) = f(x)$.

Hence from (1) we get $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx$

$= 2\int_0^a f(x)dx.$

**Case (ii)** Let $f(2a-x) = -f(x)$.

Again from (1) we get $\int_0^{2a} f(x)dx = \int_0^a f(x)dx - \int_0^a f(x)dx = 0.$

**Example 21**

Evaluate $I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

**Solution**

$I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

(1)

Also $I = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx$

$I = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$

(2)

Adding (1) and (2), $2I = \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}\,dx$

Put $\cos x = y$. Hence $\sin x\,dx = -dy$.

Now, $2I = -\pi\int_1^{-1} \frac{dy}{1+y^2} = \pi\int_{-1}^{1} \frac{dy}{1+y^2} = \pi[\tan^{-1} y]_{-1}^{1}$

$= \pi(\pi/2).$

$\therefore I = \frac{\pi^2}{4}.$

**Example 22**

Evaluate $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x}\,dx$

**Solution**

$I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} \, dx$ (1)

Also, $I = \frac{\int_0^{\frac{\pi}{2}} \sin^2\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} \, dx$

$\therefore$ Adding (1) and (2), $2I = \int_0^{\pi/2} \frac{1}{\cos x + \sin x} \, dx$

Put $t = \tan(x/2)$. Hence $dt = \frac{1}{2} \sec^2(x/2) \, dx$

$\therefore dx = \frac{2dt}{1+t^2}$

Now $I = \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$

$$= \int_0^1 \frac{2\, dt}{(1+t^2)\left[\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}\right]}$$

$$= 2 \int_0^1 \frac{dt}{1 - t^2 + 2t}$$

$$= 2 \int_0^1 \frac{dt}{\left(\sqrt{2}\right)^2 - (t-1)^2}$$

$$= \left[2\left(\frac{1}{2\sqrt{2}}\right) \log\left(\frac{\sqrt{2} + t - 1}{\sqrt{2} - t + 1}\right)\right]_0^1$$

$$= \frac{1}{\sqrt{2}}\left[\log 1 - \log\left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right)\right]$$

$$= \frac{1}{\sqrt{2}} \log\left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right)$$

$\therefore I = \frac{1}{2\sqrt{2}} \log\left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right)$

**Example 23**

Evaluate $I = \int_0^{\pi/4} \log(1 + \tan\theta) \, d\theta$.

**Solution**

$I = \int_0^{\pi/4} \log(1 + \tan\theta) \, d\theta$.

Also, $I = \int_0^{\pi/4} \log[1 + \tan(\pi/4 - \theta)] \, d\theta$.

$$= \int_0^{\pi/4} \log\left(1 + \frac{1 - \tan\theta}{1 + \tan\theta}\right) \, d\theta.$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1 + \tan\theta}\right) \, d\theta$$

$\therefore I = \int_0^{\pi/4} \log 2 \, d\theta - \int_0^{\pi/4} \log(1 + \tan\theta) \, d\theta$

$$= \int_0^{\pi/4} \log 2 \, d\theta - I.$$

$\therefore 2I = \int_0^{\pi/4} \log 2 \, d\theta = \log 2 \, [\theta]_0^{\pi/4}$

$= \frac{\pi}{4} \log 2.$

$\therefore I = \frac{\pi}{8} \log 2.$

**Exercise 6**

1. Prove that $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx = \frac{\pi}{4}$
2. Prove that $\int_0^{2} x\sqrt{(2-x)}\, dx = \frac{16\sqrt{2}}{15}$
3. Prove that $\int_0^{\pi} x \sin^3 x\, dx = \frac{2\pi}{3}$
4. Prove that $\int_0^{\pi} \frac{x}{a^2-\cos^2 x} dx = \frac{\pi^2}{2a\sqrt{(a^2-1)}}$
5. Prove that $\int_0^{\pi/2} \frac{x}{\sin x+\cos x} dx = \frac{\pi}{2\sqrt{2}} \log\left(\sqrt{2}+1\right)$

**4.7 Integration by Parts**

**Theorem 1 (Integration by parts)**

Let $u$ and $v$ be differentiable functions of $x$. Then $\int u\, dv = uv - \int v\, du$.

**Proof**

We know that $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.

Integrating we get $uv = \int u\frac{dv}{dx}\, dx + \int v\frac{du}{dx}\, dx$

$$= \int u\, dv + \int v\, du$$

$\therefore \int u\, dv = uv - \int v\, du.$

**Note** The method of evaluating a given integral by using the above theorem is called integration by parts. In applying this method we must choose $u$ and $v$ carefully so that the resulting integral is simpler than the given integral.

**Theorem 2 (Bernoulli's formula)**

Let $u$ and $v$ be differentiable functions $x$. Suppose there exists a positive integer $n$ such that $u^{(n)} = 0$ then

$\int u\, dv = uv - u'v_1 + u''v_2 - \cdots (-1)^n u^{(n)} v_n$ where $v_1 = \int v\, dx$ ; $v_2 = \int v_1 dx$ ; ...

**Proof**

$\int u\, dv = uv - \int v\, du$ (by theorem 1)

$$= uv - \int u'\, d(v_1)$$

$$= uv - u'v_1 + \int v_1\, du'$$

$$= uv - u'v_1 + \int u'' d(v_2)$$
$$= uv - u'v_1 + u''v_2 - \int v_2\, du''$$

Proceeding like this we get the required formula.

**Example 24**

Prove that $\int u \frac{d^2v}{dx^2}\, dx = u\frac{dv}{dx} - v\frac{du}{dx} + \int v \frac{d^2u}{dx^2}\, dx$

**Solution**

$$\int u \frac{d^2v}{dx^2}\, dx = \int u \frac{d}{dx}\left(\frac{dv}{dx}\right) dx$$
$$= u\frac{dv}{dx} - \int \frac{dv}{dx}\, du \qquad \text{(by integration by parts)}$$
$$= u\frac{dv}{dx} - \int \frac{dv}{dx}\frac{du}{dx}\, dx$$
$$= u\frac{dv}{dx} - \int \left(\frac{du}{dx}\right) dv$$
$$= u\frac{dv}{dx} - \left[v\frac{du}{dx} - \int v\frac{d^2u}{dx^2}\, dx\right]$$
$$= u\frac{dv}{dx} - v\frac{du}{dx} + \int v\frac{d^2u}{dx^2}\, dx$$

**Example 25**

Evaluate $\int x\, e^x dx$

**Solution**

$$\int x\, e^x dx = \int x\, d(e^x)$$
$$= x\, e^x - \int e^x dx$$
$$= x\, e^x - e^x$$
$$= e^x(x-1)$$

**Example 26**

Evaluate $\int \frac{x \sin^{-1} x}{\sqrt{(1-x^2)}}\, dx$

**Solution**

We note that $d\left[-\sqrt{(1-x^2)}\right] = \frac{x}{\sqrt{(1-x^2)}}$

$$\therefore \int \frac{x \sin^{-1} x}{\sqrt{(1-x^2)}}\, dx = \int \sin^{-1} x\ d\left[-\sqrt{(1-x^2)}\right]$$
$$= -\sqrt{(1-x^2)}\sin^{-1} x + \int dx$$
$$= x - \sqrt{(1-x^2)}\sin^{-1} x$$

**Example 27**

Prove that $\int_0^{\pi/4} \theta \sec^2 \theta \; d\theta = \frac{1}{4}(\pi - 2\log 2)$.

**Solution**

$$\int \theta \sec^2 \theta \; d\theta = \int \theta \; d(tan\,\theta)$$
$$= \theta \tan\theta - \int \tan\theta \; d\theta = \theta\tan\theta - \log\sec\theta$$
$$\therefore \int_0^{\pi/4} \theta \sec^2\theta \; d\theta = [\theta\tan\theta - \log\sec\theta]_0^{\pi/4}$$
$$= \frac{\pi}{4} - \log\sqrt{2} = \frac{\pi}{4} - \frac{1}{2}\log 2$$
$$= \frac{1}{4}(\pi - 2\log 2).$$

**Example 28**

Prove that $\int e^{ax} \cos bx \; dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$

**Solution**

Let $I = \int e^{ax}\cos bx \; dx$

$$\therefore I = \int \cos bx \; d\left(\frac{e^{ax}}{a}\right)$$
$$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\int e^{ax}\sin bx \; dx$$
$$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\left[\frac{e^{ax}}{a}\sin bx - \frac{b}{a}\int e^{ax}\cos bx \; dx\right]$$
$$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a^2}e^{ax}\sin bx - \left(\frac{b^2}{a^2}\right) I$$
$$\therefore I\left(1 + \frac{b^2}{a^2}\right) = e^{ax}\left[\frac{\cos bx}{a} + \frac{b}{a^2}\sin bx\right].$$
$$\therefore I = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

**Note** Similarly, $\int e^{ax}\sin bx \; dx = \frac{e^{ax}(a\cos bx - b\sin bx)}{a^2+b^2}$

**Exercise 7**

Integrate the following with respect to $x$.

1. $(px+q)\,e^{ax+b}$
2. $\frac{\log x}{(x+1)^2}$
3. $\tan^{-1}(\sqrt{x})$
4. $x\cos^2 x$
5. $\sin^{-1}\sqrt{\left(\frac{x}{x+1}\right)}$
6. $\frac{e^x(x+1)}{(x+2)^2}$

**Answers**

1. $\frac{e^{ax+b}}{a^2}(apx + aq - p)$
2. $\frac{x}{x+1}\log x - \log(x+1)$

3. $(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}$ 4. $\frac{1}{4}x^2+\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x$

5. $x\sin^{-1}\sqrt{\left(\frac{x}{x+1}\right)}-\sqrt{x}+\tan^{-1}\sqrt{x}$ 6. $\frac{e^x}{x+2}$

## 4.8 Reduction formulae

**I. Establish a reduction formulae for $\int x^n e^{ax}\, dx$ where $n \in N$.**

**Proof**

$I_n = \int x^n e^{ax}\, dx$

$I_n = \int x^n\, d\left(\frac{e^{ax}}{a}\right) = x^n\left(\frac{e^{ax}}{a}\right) - \left(\frac{n}{a}\right)\int e^{ax}\, x^{n-1} dx$

$= \frac{x^n e^{ax}}{a} - \left(\frac{n}{a}\right) I_{n-1}$

The reduction formula for $I_n$ is $I_n = \frac{x^n e^{ax}}{a} - \left(\frac{n}{a}\right) I_{n-1}$.

The ultimate integral is $I_0 = \int e^{ax}\, dx = \frac{e^{ax}}{a}$.

**2. Reduction formula for $I_n = \int x^n \cos ax\ dx$ where $n \in N$.**

**Proof**

Let $I_n = \int x^n \cos ax\ dx$

$= \int x^n\, d\left(\frac{\sin ax}{a}\right) = x^n\left(\frac{\sin ax}{a}\right) - \left(\frac{n}{a}\right)\int x^{n-1}\sin ax\ dx$

$= x^n\left(\frac{\sin ax}{a}\right) - \left(\frac{n}{a}\right)\left[\left(-\frac{\cos ax}{a}\right)x^{n-1} + \frac{n-1}{a}\int x^{n-2}\cos ax\ dx\right]$

$= \left(\frac{x^n\ \sin ax}{a}\right) - \left(\frac{n}{a^2}\right)x^{n-1}\cos ax - \frac{n(n-1)}{a^2} I_{n-2}$

The ultimate integral depends on $n$ being odd or even.

**Case i.** n is odd. Then the ultimate integral reduces to

$I_1 = \int x\cos ax\, dx = \frac{x\sin ax}{a} - \frac{1}{a}\int \sin ax\ dx$

$= \frac{x\sin ax}{a} + \frac{\cos ax}{a^2}$

**Case ii.** n is even. Then the ultimate integral reduces to

$I_0 = \int \cos ax\ dx = \frac{\sin ax}{a}$

**3. Reduction formula $I_n = \int \sin^n x\ dx$. $(n \in N)$ & find $\int_0^{\pi/2} \sin^n x\ dx$**

**Proof**

$I_n = \int \sin^n x\ dx$

$I_n = \int \sin^{n-1} x\sin x\, dx = \int \sin^{n-1} x\ d(-\cos x)$

$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\, dx$

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)dx$

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^{n} x\,dx$

$\therefore I_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2} - (n-1)I_n$

$\therefore n\,I_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2}$

The ultimate integral is $= \begin{cases} I_1 = \int \sin x\ dx & if\ n\ is\ odd \\ I_0 = \int dx & if\ n\ is\ even \end{cases}$

$= \begin{cases} -\cos x & if\ n\ is\ odd \\ x & if\ n\ is\ even \end{cases}$

Now $I_n = -\frac{\sin^{n-1} x \cos x}{n} + \left(\frac{n-1}{n}\right) I_{n-2}$

$\int_0^{\pi/2} \sin^n x\ dx = \left[-\frac{\sin^{n-1} x \cos x}{n}\right]_0^{\pi/2} + \left(\frac{n-1}{n}\right)\int_0^{\pi/2} \sin^{n-2} x\ dx$

$= \left(\frac{n-1}{n}\right)\int_0^{\pi/2} \sin^{n-2} x\ dx$

$= \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\int_0^{\pi/2} \sin^{n-4} x\ dx$

$= \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right) \ldots\ldots\ldots U_n$ where $U_n$ is the ultimate integral which depends upon $n$ being odd or even.

**Case (i)** If $n$ is even $U_0 = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \pi/2$.

**Case (ii)** If $n$ is odd $U_1 = \int_0^{\pi/2} \sin x\ dx = [-\cos x]_0^{\pi/2} = 1$

$$\int_0^{\pi/2} \sin^n x\ dx = \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\ldots\frac{1}{2}\left(\frac{\pi}{2}\right) & if\ n\ is\ even \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\ldots\frac{2}{3}(1) & if\ n\ is\ odd \end{cases}$$

**Note** Similarly we can establish a reduction formula for $\int \cos^n x\ dx$ and hence evaluate $\int_0^{\pi/2} \cos^n x\ dx$ by means of the formula given below.

If $I_n = \int \cos^n x\ dx$ then $n\,I_n = \cos^{n-1} x \sin x + (n-1)I_{n-2}$

$$\int_0^{\pi/2} \cos^n x\ dx = \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\ldots\frac{1}{2}\left(\frac{\pi}{2}\right) & if\ n\ is\ even \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\ldots\frac{1}{3}(1) & if\ n\ is\ odd \end{cases}$$

**4. Establish a reduction formula for $I_n = \int \tan^n x\ dx$**

**Proof**

$I_n = \int \tan^n x\ dx$

$= \int \tan^{n-2} x \tan^2 x\ dx$

$= \int \tan^{n-2} x\,(\sec^2 x - 1)\,dx$

$= \int \tan^{n-2} x \sec^2 x\ dx - \int \tan^{n-2} x\,dx$

$= \int \tan^{n-2} x\,d(\tan x) - I_{n-2}$

$\therefore I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$

The ultimate integral is given by

$I_0 = \int dx = x \;\; if\ n\ is\ even.$

$I_1 = \int \tan x \;\, dx = \log \sec x \;\;\; if\ n\ is\ odd.$

**5. Establish a reduction formula for $I_n = \int \cot^n x \;\, dx$**

**Proof**

$I_n = \int \cot^{n-2} x \cot^2 x \;\, dx$

$\quad = \int \cot^{n-2} x\, (\operatorname{cosec}^2 x - 1)\, dx$

$\quad = -\int \cot^{n-2} x \;\, d(\cot x) - \int \cot^{n-2} x \;\, dx$

$\therefore I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$

The ultimate integral is given by

$I_0 = \int dx = x \;\; if\ n\ is\ even.$

$I_1 = \int \cot x \;\, dx = \log \sin x \;\;\; if\ n\ is\ odd.$

**6. Establish a reduction formula for $I_n = \int \sec^n x \;\, dx$**

**Proof**

$I_n = \int \sec^n x \;\, dx$

$\quad = \int \sec^{n-2} x \;\, d(\tan x)$

$\quad = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x \;\, dx$

$\quad = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x\, (\sec^2 x - 1)\, dx$

$\quad = \sec^{n-2} x \tan x - (n-2)[I_n - I_{n-2}].$

$\therefore (n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$

The ultimate integral is given by

$I_0 = \int dx = x \;\; if\ n\ is\ even.$

$I_1 = \int \sec x \;\, dx = \log(\sec x + \tan x) \;\;\; if\ n\ is\ odd.$

**7. Establish a reduction formula for $I_n = \int \operatorname{cosec}^n x \;\, dx$**

**Proof**

$I_n = \int \operatorname{cosec}^n x \;\, dx$

$\quad = -\int \operatorname{cosec}^{n-2} x \;\, d(\cot x)$

$\quad = -\operatorname{cosec}^{n-2} x \cot x - (n-2) \int \operatorname{cosec}^{n-2} x\, (\operatorname{cosec}^2 x - 1)\, dx$

$I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2)[I_n - I_{n-2}].$

$\therefore (n-1)I_n = - - \operatorname{cosec}^{n-2} x \cot x - (n-2)I_{n-2}$

The ultimate integral is given by

$I_0 = \int dx = x \;\; if\ n\ is\ even.$
$I_1 = \int \operatorname{cosec} x \;\; dx = -\log(\operatorname{cosec} x + \cot x) \;\; if\ n\ is\ odd.$

**8. Establish a reduction formula for $I_{m,n} = \int \sin^m x \cos^n x \;\; dx$ where $m, n \geq 1$.**

**Proof**

$$I_{m,n} = \int \cos^{n-1} x\,(\sin^m x \cos x \;\; dx)$$
$$= \int (\cos x)^{n-1} d\left(\frac{(\sin x)^{m+1}}{m+1}\right)$$
$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{m+2} x \cos^{n-2} x \;\; dx$$
$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{m} x \cos^{n-2} x\,(1-\cos^2 x)\,dx$$
$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\left[I_{m,n-2} - I_{m,n}\right]$$
$$\therefore \left[1 - \frac{n-1}{m+1}\right] I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \left(\frac{n-1}{m+1}\right) I_{m,n-2}.$$
$$(m+n) I_{m,n} = \cos^{n-1} x \sin^{m+1} x + (n-1) I_{m,n-2}$$
$$\therefore I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \left(\frac{n-1}{m+n}\right) I_{m,n-2}.$$

**Note** By reducing the power of $\sin x$ we may arrive at an alternative reduction formula given by $I_{m,n} = \frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \left(\frac{m-1}{m+n}\right) I_{m-2,n}$
The ultimate integral depends on $m$ and $n$ being odd or even.

**9. Evaluate $\int_0^{\pi/2} \sin^m x \cos^n x \;\; dx$.**

**Proof**

We have proved that $I_{m,n} = \int \sin^m x \cos^n x \;\; dx$
$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \left(\frac{n-1}{m+n}\right) I_{m,n-2}$$

Let $f(m,n) = \int_0^{\pi/2} \sin^m x \cos^n x \;\; dx$

$$\therefore f(m,n) = \left[\frac{\cos^{n-1} x \sin^{m+1} x}{m+n}\right]_0^{\pi/2} + \left(\frac{n-1}{m+n}\right) f(m, n-2).$$
$$\therefore f(m,n) = \left(\frac{n-1}{m+n}\right) f(m, n-2).$$

**Case (i)** m is even and n is even.

Let $m = 2p$ and $n = 2q$ where $p, q \in N$.

$$\therefore f(2p, 2q) = \left(\frac{2q-1}{2p+2q}\right) f(2p, 2q-2)$$

$$= \frac{(2q-1)(2q-3)}{(2p+2q)(2p+2q-2)} f(2p, 2q-4)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$= \frac{(2q-1)(2q-3)\dots 1}{(2p+2q)(2p+2q-2)\dots(2p+2)} f(2p, 0)$$

Now $f(2p, 0) = \int_0^{\pi/2} \cos^{2p} x \ dx$

$$= \left(\frac{2p-1}{2p}\right)\left(\frac{2p-3}{2p-2}\right)\dots\frac{1}{2}\left(\frac{\pi}{2}\right).$$

Hence $f(2p, 2q) = \frac{[1.3.5\dots(2p-1)][1.3.5\dots(2q-1)]}{2.4.6\dots(2p+2q)}\left(\frac{\pi}{2}\right)$

**Case (ii)** m is odd and n is even.

Let $m = 2p - 1$ and $n = 2q$ where $p, q \in N$.

$\therefore f(2p-1, 2q) = \left(\frac{2q-1}{(2p+2q-1)}\right) f(2p-1, 2q-2)$

$$= \frac{(2q-1)(2q-3)}{(2p+2q-1)(2p+2q-3)} f(2p-1, 2q-4)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$= \frac{(2q-1)(2q-3)\dots 3.1}{(2p+2q-1)(2p+2q-3)\dots(2p+1)} f(2p-1, 0)$$

Now $f(2p-1, 0) = \int_0^{\pi/2} \sin^{2p-1} x \ dx$

$$= \left(\frac{2p-2}{2p-1}\right)\left(\frac{2p-4}{2p-3}\right)\dots\frac{2}{3}(1).$$

$\therefore f(2p-1, 2q) = \frac{[2.4\dots(2p-2)][1.3.5\dots(2q-1)]}{1.3.5\dots(2p+2q-1)}$

**Case (iii)** m is even and n is odd.

Let $m = 2p$ and $n = 2q - 1$. We can prove that

$\therefore f(2p, 2q-1) = \frac{[1.3.5\dots(2p-1)][2.4.6\dots(2q-2)]}{1.3.5\dots(2p+2q-1)}$

**Case (iv)** m is odd and n is odd.

Let $m = 2p - 1$ and $n = 2q - 1$ where $p, q \in N$

$\therefore f(2p-1, 2q-1) = \left(\frac{2q-2}{(2p+2q-2)}\right) f(2p-1, 2q-3)$

$$= \frac{(2q-2)(2q-4)}{(2p+2q-1)(2p+2q-3)} f(2p-1, 2q-5)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$= \frac{(2q-2)(2q-4)\dots 2}{(2p+2q-2)(2p+2q-4)\dots(2p+2)} f(2p-1, 1)$$

Now $f(2p-1, 1) = \int_0^{\pi/2} \sin^{2p-1} x \cos x \ dx$

$$= \left[\frac{\sin^{2p} x}{2p}\right]_0^{\pi/2} = \frac{1}{2p}$$

$\therefore f(2p-1, 2q-1) = \frac{[2.4.6\dots(2p-2)][2.4.6\dots(2q-2)]}{2.4.6\dots(2p+2q-2)}$

**Example**

1. $\int_0^{\pi/2} \sin^6 x \cos^8 x \; dx = \frac{1.3.5.1.3.5.7}{2.4.6.8.10.12.14}\left(\frac{\pi}{2}\right) = \frac{5\pi}{2^{12}}$
2. $\int_0^{\pi/2} \sin^6 x \cos^5 x \; dx = \frac{1.3.5.2.4}{1.3.5.7.9.11} = \frac{8}{693}$
3. $\int_0^{\pi/2} \sin^5 x \cos^6 x \; dx = \frac{2.4.1.3.5}{1.3.5.7.9.11} = \frac{8}{693}$
4. $\int_0^{\pi/2} \sin^5 x \cos^5 x \; dx = \frac{2.4.2.4}{2.4.6.8.10} = \frac{1}{60}$

**Example 29**

Evaluate $I = \int_0^1 x^2(1-x^2)^{3/2}\, dx$

**Solution**

Put $x = \sin\theta$. Hence $dx = \cos\theta \; d\theta$.

When $x = 0, \theta = 0$ and when $x = 1, \theta = \pi/2$.

$$\therefore I = \int_0^{\pi/2} \sin^2\theta\,(\cos^2\theta)^{3/2}\cos\theta\, d\theta$$
$$= \int_0^{\pi/2} \sin^2\theta \cos^4\theta\, d\theta$$
$$= \frac{1.3.1}{6.4.2}\left(\frac{\pi}{2}\right)$$
$$= \frac{\pi}{32}.$$

**Example 30**

Establish a reduction formula for $\int x^m (\log x)^n \, dx$

**Solution**

Let $I_{m,n} = \int x^m (\log x)^n \, dx$

$$\therefore I_{m,n} = \frac{1}{m+1}\int (\log x)^n \, d(x^{m+1})$$
$$= \frac{1}{m+1}\left[(\log x)^n x^{m+1} - \int x^{m+1}\, n\,(\log x)^{n-1}\left(\frac{1}{x}\right) dx\right]$$
$$= \frac{1}{m+1}\left[(\log x)^n x^{m+1} - n\int x^m\,(\log x)^{n-1} dx\right]$$
$$= \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1} I_{m,n-1}$$

The ultimate integral is $I_{m,0} = \int x^m dx = \frac{x^{m+1}}{m+1}$.

**Example 31**

If $I_n = \int_0^{\pi/2} \theta \sin^n\theta \, d\theta$ and $n > 1$ prove that $I_n = \left(\frac{n-1}{n}\right) I_{n-2} + \frac{1}{n^2}$. Hence deduce that $I_5 = \frac{149}{225}$.

**Solution**

$I_n = \int (\sin\theta)^{n-1}(\theta \sin\theta)d\theta$

Taking $u = (\sin\theta)^{n-1}$ and $dv = \theta \sin\theta \, d\theta$ we get

$v = -\theta\cos\theta + \sin\theta$

$$\therefore I_n = [(\sin\theta - \theta\cos\theta)(\sin\theta)^{n-1}]_0^{\frac{\pi}{2}}$$
$$-(n-1)\int_0^{\pi/2}(\sin\theta - \theta\cos\theta)(\sin\theta)^{n-2}\cos\theta \, d\theta$$

$$= 1 - (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-1}\theta\cos\theta\, d\theta$$
$$+(n-1)\int_0^{\frac{\pi}{2}}\theta(\sin\theta)^{n-2}\cos^2\theta\, d\theta$$

$$= 1 - (n-1)\int_0^{\frac{\pi}{2}}(\sin\theta)^{n-1}d(\sin\theta)$$
$$+(n-1)\int_0^{\frac{\pi}{2}}\theta(\sin\theta)^{n-2}(1-\sin^2\theta)d\theta$$

$$= 1 - \frac{n-1}{n}[\sin^n\theta]_0^{\pi/2} + (n-1)I_{n-2} - (n-1)I_n$$

$$\therefore I_n(1+n-1) = 1 - \left(\frac{n-1}{n}\right) + (n-1)I_{n-2}$$

$$\therefore nI_n = \frac{1}{n} + (n-1)I_{n-2}$$

$$\therefore I_n = \left(\frac{n-1}{n}\right)I_{n-2} + \frac{1}{n^2}.$$

Now, $I_5 = \left(\frac{4}{5}\right)I_3 + \frac{1}{25}$

$I_3 = \left(\frac{2}{3}\right)I_1 + \frac{1}{9}$

Also, $I_1 = \int_0^{\pi/2}\theta\sin\theta\, d\theta$

$$= [-\theta\cos\theta]_0^{\pi/2} + \int_0^{\pi/2}\cos\theta\, d\theta$$
$$= [-\theta\cos\theta + \sin\theta]_0^{\pi/2} = 1$$

Hence $I_3 = \frac{2}{3} + \frac{1}{9} = \frac{7}{9}$

$$\therefore I_5 = \frac{4}{5}\left(\frac{7}{9}\right) + \frac{1}{25} = \frac{149}{225}.$$

**Exercise 8**

Evaluate the following integrals

1. $\int_0^{\pi/2} x^2 \sin x\, dx$
2. $\int_0^{\pi/4} \sec^5 x\, dx$
3. $\int_0^1 x^2(1-x^2)^{5/2}dx$
4. $\int_0^{\pi/4} \tan^3 x\, dx$
5. $\int_0^{\pi/2} \sin^2 x\,(\sin^3 x + \cos^3 x)dx$

6. If $I_n = \int_0^{\pi/2} x\cos^n x\, dx$ and $n > 1$ show that $I_n = -\frac{1}{n^2} + \left(\frac{n-1}{n}\right) I_{n-2}$. Hence prove that $I_4 = \frac{3\pi^2}{64} - \frac{1}{4}$.

**Answers**

1. $\pi - 2$ 2. $\frac{7\sqrt{2}}{8} + \frac{3}{8}\log\left(\sqrt{2}+1\right)$ 3. $\frac{\pi}{32}$

4. $\frac{1}{2} - \frac{1}{2}\log 2$ 5. $\frac{2}{3}$

## 4.9 Integration as the limit of a sum

We know that $\int_a^b f(x)dx = \begin{cases} \lim\limits_{h\to 0} h\sum_{i=1}^{n} f(a+ih) \\ \lim\limits_{h\to 0} h\sum_{i=1}^{n} f[a+(i-h)] \end{cases}$ where $h = \frac{b-a}{n}$.

In particular if $a = 0$ and $b = 1$ we have

$\int_0^1 f(x)dx = \lim\limits_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)$

We use this formula to express some definite integrals as the limit of a sum and also to evaluate certain limits.

**Example 32**

Evaluate $\int_a^b e^x dx$

**Solution**

$$\begin{aligned}\int_a^b e^x dx &= \lim_{n\to\infty} h \sum_{i=1}^{n} f[a+(i-1)h] \\ &= \lim_{n\to\infty} h[f(a) + f(a+h) + \cdots + f(a+(n-1)h)] \\ &= \lim_{n\to\infty} h\left[e^a + e^{a+h} + \cdots + e^{a+(n-1)h}\right] \\ &= \lim_{n\to\infty} he^a\left[1 + e^h + e^{2h} + \cdots + e^{(n-1)h}\right] \\ &= \lim_{n\to\infty} he^a\left[\frac{e^{nh}-1}{e^h-1}\right] \\ &= \lim_{n\to\infty} e^a(e^{b-a}-1)\left(\frac{h}{e^h-1}\right) \qquad \left(since\ h = \frac{b-a}{n}\right) \\ &= e^a(e^{b-a}-1) \qquad \left(since\ \lim_{n\to\infty}\left(\frac{h}{e^h-1}\right) = 1\right) \\ &= e^b - e^a\end{aligned}$$

**Example 33**

Evaluate $\int_a^b \sin x\, dx$

**Solution**

$$\int_a^b f(x)dx = \lim_{n\to\infty} h \sum_{i=1}^{n} f[a+(i-1)h]$$
$$= \lim_{n\to\infty} h \sum_{i=1}^{n} \sin[a+(i-1)h]$$
$$= \lim_{n\to\infty} h\,[\sin a + \sin(a+h) + \cdots + \sin(a+(n-1)h)]$$
$$= \lim_{n\to\infty} h \frac{\sin(nh/2)}{\sin(h/2)} \sin\left[a+(n-1)\frac{h}{2}\right]$$
$$= \lim_{n\to\infty} \frac{(h/2)}{\sin(h/2)} 2\sin\left[\frac{1}{2}(b-a)\right] \sin\left[a+\frac{b-a}{2}-\frac{h}{2}\right]$$
$$= 2\sin\left[\frac{1}{2}(b-a)\right]\sin\left[\frac{1}{2}(a+b)\right]$$
$$= \cos a - \cos b.$$

**Example 34**

Express $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{\sqrt{(n^2-i^2)}}$ as a definite integral and hence evaluate the limit.

**Solution**

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{\sqrt{(n^2-i^2)}} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{(1-i/n)^2}}$$
$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \text{ where } f(x) = \frac{1}{\sqrt{(1-x^2)}}$$
$$= \int_0^1 f(x)dx$$
$$= \int_0^1 \frac{dx}{\sqrt{(1-x^2)}} = [\sin^{-1} x]_0^1 = \frac{\pi}{2}.$$

**Example 35**

Express $\lim_{n\to\infty} \left[\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n}\right]$ as a definite integral and hence evaluate the limit.

Solution

$$\lim_{n\to\infty} \sum_{i=1}^{2n} \frac{1}{n+i} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{2n} \frac{1}{1+\left(\frac{i}{n}\right)} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{2n} f\left(\frac{i}{n}\right) \text{ where } f(x) = \frac{1}{1+x}$$
$$= \int_0^2 f(x)dx$$
$$= \int_0^2 \frac{1}{1+x} dx = [\log(1+x)]_0^2$$
$$= \log 3.$$

**Exercise 9**

1. Evaluate the following by expressing it as the limit of a sum

(i) $\int_a^b x^2 dx$ (ii) $\int_0^1 x^3 dx$ (iii) $\int_a^b \cos x\, dx$

2. Express the following as a definite integral and hence evaluate

(i) $\lim_{n\to\infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right]$

(ii) $\lim_{n\to\infty} \sum_{i=0}^{n} \frac{i^3}{i^4+n^4}$

(iii) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n^2}{(n+i)^3}$

**Answers**

1. (i) $\frac{1}{3}(b^3 - a^3)$ (ii) $\frac{1}{4}$ (iii) $\sin b - \sin a$

2. (i) $\log 2$ (ii) $\frac{1}{4}\log 2$ (iii) $\frac{3}{8}$

## UNIT-V

## DOUBLE AND TRIPLE INTEGRALS

### 5.0 Introduction

In this section we shall discuss the Double integral, Evaluation of double integral, triple integral, change of variables in double and triple integral and introduce two important functions defined in terms of some improper integrals and derive some properties of these functions and Fourier series.

### 5.1 Double integrals

**Definition**

Let $f(x,y)$ be a bounded real valued function defined on a closed rectangle $R = \{(x,y) |\ a \le x \le b \ and \ c \le y \le d\}$.

Let $P$ be a partition of $R$ into $m$ sub-rectangles $R_1, R_2, \dots, R_m$ by lines parallel to coordinates axes. We define the norm of the partition $P$ as

$\|P\| =$ maximum of the lengths of the diagonals of the sub-rectangles $R_1, R_2, \dots, R_m$. Let $(x_i, y_i) \in R_i$.

Consider the sum $\sum_{i=1}^{m} f(x_i, y_i) A(R_i)$ where $A(R_i)$ stands for the area of the rectangle $R_i$.

The function $f(x,y)$ is said to be Riemann integrable over $R$ if $\lim_{\|P\| \to 0} \sum_{i=1}^{m} f(x_i, y_i) A(R_i)$ exists and is finite.

The value of the above limit is called the double integral of $f(x,y)$ over $R$ and it is denoted by $\iint_R f(x,y)\, dx\, dy$.

Thus $\iint_R f(x,y)\, dx\, dy = \lim_{\|P\| \to 0} \sum_{i=1}^{m} f(x_i, y_i) A(R_i)$.

Now, let $f(x,y)$ be a function defined on a bounded set $D$. Let $R$ be any closed rectangle with sides parallel to the coordinate axes containing $D$. We define a new function $f_D$ on $R$ by

$$f_D(x,y) = \begin{cases} f(x,y) & if\ (x,y) \in D \\ 0 & if\ (x,y) \notin D \end{cases}$$

$f(x,y)$ is said to be integrable over $D$ iff $f_D(x,y)$ is integrable over $R$.

The double integral of $f(x,y)$ over $D$ is defined by the equation

$$\iint_D f(x,y)\, dx\, dy = \iint_R f_D(x,y)\, dx\, dy$$

This definition is independent of the choice of the rectangle $R$ containing $D$. The double integral $\iint_D f(x,y)\,dx\,dy$ is also written as $\iint_D f(x,y)\,dA$ or $\iint_D f(x,y)\,d(x,y)$.

**Note** $\iint_D dx\,dy$ represents the area of the region $D$.

## 5.2 Evaluation of double integrals

### Double integrals

Let $f(x,y)$ be a continuous function defined on a closed rectangle $R = \{(x,y)|\ a \le x \le b \ and\ c \le y \le d\}$.

For any fixed $x \in [a,b]$ consider the integral $\int_c^d f(x,y)dy$.

The value of this integral depends on $x$ and we get a new function of $x$. This can be integrated with respect to $x$ and we get $\int_a^b \left[\int_c^d f(x,y)dy\right] dx$. This is called an iterated integral.

Similarly we can define another integral $\int_c^d \left[\int_a^b f(x,y)dx\right] dy$.

For continuous functions $f(x,y)$ we have $\iint_R f(x,y)\,dx\,dy = \int_a^b \left[\int_c^d f(x,y)dy\right] dx = \int_c^d \left[\int_a^b f(x,y)dx\right] dy$

We omit the proof of this result.

If $f(x,y)$ is continuous on a bounded region $S$ and if $S$ is given by $S = \{(x,y)|\ a \le x \le b \ and\ \varphi_1(x) \le y \le \varphi_2(x)\}$ where $\varphi_1$ and $\varphi_2$ are two continuous functions defined on $[a,b]$ then

$\iint_S f(x,y)\,dx\,dy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\right] dx.$

The iterated integral in the right hand side is also written in the form

$\int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy.$

Similarly if $S = \{(x,y)|\ c \le y \le d \ and\ \varphi_1(y) \le x \le \varphi_2(y)\}$ then $\int_c^d \left[\int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)dx\right] dy$

If $S$ cannot be written in either of the above two forms we divide $S$ into finite number of subregions such that each of the subregion can be represented in one of the above forms and we get the double integral over $S$ by adding the integrals over these subregions.

Hence to evaluate $\iint_D f(x,y)\,dx\,dy$ we first convert it to an iterated integral of the two forms given above.

**Example 1**

Evaluate $I = \int_0^{4a} \int_{x^2/2a}^{2\sqrt{(ax)}} xy\,dy\ dx$

**Solution**

$$I = \int_0^{4a} \left[\frac{xy^2}{2}\right]_{x^2/4a}^{2\sqrt{(ax)}} dx.$$

$$= \frac{1}{2}\int_0^{4a} x\ \left[4ax - \frac{x^4}{16a^2}\right] dx = \frac{1}{2}\left[\frac{4ax^3}{3} - \frac{x^6}{96a^2}\right]_0^{4a}$$

$$= \frac{64\,a^4}{3}.$$

**Example 2**

Evaluate $I = \int_0^{a} dx \int_0^{b\sqrt{[1-(x^2/a^2)]}} x^3 y\,dy$

**Solution**

$$I = \int_0^{a} \left[\frac{1}{2}x^3y^2\right]_0^{b\sqrt{[1-(x^2/a^2)]}} dx$$

$$= \frac{1}{2}\int_0^{a} b^2x^3\left(1 - \frac{x^2}{a^2}\right) dx = \frac{1}{2}b^2\left[\frac{1}{4}x^4 - \frac{x^6}{6a^2}\right]_0^{a}$$

$$= \frac{a^4b^2}{24}.$$

**Example 3**

Evaluate $I = \int_0^{\pi} \int_0^{a\cos\theta} r\sin\theta\ dr\,d\theta$

**Solution**

$$I = \int_0^{\pi} \sin\theta \left[\frac{1}{2}r^2\right]_0^{a\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2\cos^2\theta\sin\theta\,d\theta = -\frac{1}{2}a^2\int_0^{\pi}\cos^2\theta\,d(\cos\theta)$$

$$= -\frac{1}{6}a^2[\cos^3\theta]_0^{\pi}$$

$$= \frac{1}{3}a^2.$$

**Example 4**

Evaluate $\int_0^{\pi/2} \int_0^{\infty} \frac{r}{(r^2+a^2)^2}\,dr\,d\theta$

**Solution**

Let $I = \int_0^{\pi/2} \int_0^{\infty} \frac{r}{(r^2+a^2)^2} dr\, d\theta$

$= \int_0^{\pi/2} \left[ \int_0^{\infty} \frac{\frac{1}{2} d(r^2)}{(r^2+a^2)^2} \right] d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2} \left[ \frac{-1}{r^2+a^2} \right]_0^{\infty} d\theta$

$= \frac{1}{2} \int_0^{\pi/2} \frac{d\theta}{a^2} = \left[ \frac{\theta}{2a^2} \right]_0^{\pi/2}$

$= \frac{\pi}{4a^2}$

**Example 5**

Evaluate $\iint_D x^2 y^2 dx\, dy$ where D is the circular disc $x^2 + y^2 \leq 1$

**Solution**

In D, x varies from -1 to 1. For a fixed x, y varies from $-\sqrt{(1-x^2)}$ to $\sqrt{(1-x^2)}$

$\therefore \iint_D x^2 y^2\, dx\, dy = \int_{-1}^{1} \int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} x^2\, y^2\, dy\, dx$

$= 4 \int_0^1 \int_0^{\sqrt{(1-x^2)}} x^2\, y^2\, dy\, dx$

$= 4 \int_0^1 \left[ \frac{1}{3} x^2\, y^3 \right]_0^{\sqrt{(1-x^2)}} dx$

$= \frac{4}{3} \int_0^1 x^2 (1-x^2)^{3/2}\, dx$

$= \frac{4}{3} \int_0^{\pi/2} \sin^2 \theta \cos^4 \theta\, d\theta$ (putting $x = \sin \theta$)

$= \frac{4}{3} \left( \frac{1.3.1}{2.4.6} \right) \left( \frac{\pi}{2} \right) = \frac{\pi}{24}$

**Example 6**

Change the order of integration in the integral $I = \int_1^4 \int_{y/2}^{y} f(x,y) dx\, dy$

**Solution**

The region of integration $D$ is bounded by the lines $x = \frac{y}{2}; x = y; y = 1$ and $y = 4$. The region is a quadrilateral as shown in the figure.

In this region $x$ varies from $\frac{1}{2}$ to 4

When $\frac{1}{2} \leq x \leq 1$, $y$ varies from 1 to $2x$.

When $1 \leq x \leq 2$, $y$ varies from $x$ to $2x$.

When $2 \leq x \leq 4$, $y$ varies from $x$ to 4.

Hence for changing the order of integration we must divide $D$ into sub regions $D_1, D_2, D_3$ as shown in the figure

$\therefore I = \iint_D f(x,y)dx\,dy$

$= \int_{D_1}\int f(x,y)dx\,dy + \int_{D_2}\int f(x,y)dx\,dy + \int_{D_3}\int f(x,y)dx\,dy$

$= \int_{1/2}^{1}\int_{1}^{2x} f(x,y)dy\,dx + \int_{1}^{2}\int_{x}^{2x} f(x,y)dy\,dx + \int_{2}^{4}\int_{x}^{4} f(x,y)dy\,dx$

**Example 7**

Change the order of integration for $I = \int_0^{\pi/2}\int_0^{2a\cos\theta} f(r,\theta)r\,dr\,d\theta$

**Solution**

We know that $r = 2a\cos\theta$ represents a circle with centre $(a,0)$ and radius $a$.

Since $0 \leq \theta \leq \pi/2$ the region of integration is the semicircular disc lying in the first quadrant.

In this region $r$ varies from 0 to $2a$.

Further $r = 2a\cos\theta$ implies $\theta = \cos^{-1}\left(\frac{r}{2a}\right)$.

Hence for each fixed $r, \theta$ varies from 0 to $\cos^{-1}\left(\frac{r}{2a}\right)$.

Hence $I = \int_0^{2a}\int_0^{\cos^{-1}\left(\frac{r}{2a}\right)} f(r,\theta)rd\theta\;dr$.

**Example 8**

Evaluate $\iint_D (x^2+y^2)dx\,dy$ where $D$ is the region bounded by $y = x^2, x = 2$ and $y = 1$.

**Solution**

The region of integration is as shown in the figure.

In this region $x$ varies from 1 to 2 and for each fixed $x, y$ varies from 1 to $x^2$.

$\therefore \iint_D (x^2+y^2)dx\,dy = \int_1^2\int_1^{x^2}(x^2+y^2)dy\,dx$

$= \int_1^2\left[x^2y + \frac{1}{3}y^3\right]_1^{x^2} dx$

$= \int_1^2\left(x^4 + \frac{1}{3}x^6\right)dx$

$= \left[\frac{1}{5}x^5 + \frac{1}{21}x^7\right]_1^2 = \frac{1286}{105}$

**Exercise 1**

1. Evaluate the following integrals.

(i) $\int_0^1 \int_1^2 (x^2 + y^2) dx\, dy$ (ii) $\int_3^4 \int_1^2 \frac{dxdy}{(x+y)^2}$

(iii) $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^3 dy\, dx$ (iv) $\int_0^a \int_{y-a}^{2y} xydx\, dy$

(v) $\int_0^{\pi/2} \int_0^{\infty} \frac{r\, dr\, d\theta}{(r^2+a^2)^2}$ (vi) $\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2\, dr\ \ d\theta$

(vii) $\int_0^{\pi/6} \int_{\cos 2\theta}^{1} \frac{dr\, d\theta}{\sqrt{2+2r}}$

2. Change the order of integration in the following integrals

(i) $\int_1^2 \int_3^4 f(x,y) dy\, dx$

(ii) $\int_0^a \int_0^{\sqrt{ax}} x^2 dy\, dx$. Hence evaluate

(iii) $\int_0^a \int_{y^2/a}^{y} \frac{y\, dxdy}{(a-x)\sqrt{(ax-y^2)}}$. Hence evaluate

(iv) $\int_0^2 \int_{y^2}^{4y\sqrt{2}} f(x,y) dy\, dx$ (v) $\int_0^a \int_{mx}^{lx} f(x,y) dx\, dy$

3. Evaluate $\iint_D (1 + x + y) dx\, dy$ where $D$ is the region bounded by the lines $y = -x, x = \sqrt{y}, y = 0$ and $y = 2$.

**Answers**

1. (i) $\frac{8}{3}$ (ii) $\log\left(\frac{25}{24}\right)$ (iii) $\frac{2}{15}a^4$ (iv) $\frac{11}{24}a^4$

(v) $\frac{\pi}{4a^2}$ (vi) $\frac{32}{9}$ (vii) $\frac{\pi}{3} - 1$

2. (i) $\int_3^4 \int_1^2 f(x,y) dx\, dy$ (ii) $\frac{2}{7}a^4$

(iii) $\int_0^a \int_x^{\sqrt{ax}} \frac{y\, dy\, dx}{(a-x)\sqrt{(ax-y^2)}}; \frac{1}{2}\pi a$

(iv) $\int_0^4 \int_{x/4\sqrt{2}}^{\sqrt{x}} f(x,y) dy\, dx + \int_4^{8\sqrt{2}} \int_{x/4\sqrt{2}}^{2} f(x,y) dy\, dx$

(v) $\int_0^{ma} \int_{y/l}^{y/m} f(x,y) dy\, dx + \int_{ma}^{la} \int_{y/l}^{a} f(x,y) dy\, dx$

3. $\frac{44}{15}\sqrt{2} + \frac{5}{3}$

**5. 3 Triple Integrals**

The definition of triple integrals for a function $f(x,y,z)$ defined over a region $D$ in $R^3$ is analogous to the definition of double integral is defined in 5.1. In definition of 5.1 we replace rectangles by parallelepipeds and area by volume to obtain the corresponding definition of triple integrals.

A triple integral of a function defined over a region $D$ is denoted by $\iiint_D f(x,y,z)dx\,dy\,dz$ or $\iiint_D f(x,y,z)dV$ or $\iiint_D f(x,y,z)d(x,y,z)$

The triple integral can be expressed as an iterated integrals in several ways. For example if a region $D$ in $R^3$ is given by

$D = \{(x,y,z)|a \le x \le b;\ \Phi_1(x) \le y \le \Phi_2(x);\ \psi_1(x,y) \le z \le \psi_2(x,y)\}$

then $\iiint_D f(x,y,z)dx\,dy\,dz = \int_a^b \int_{\Phi_1(x)}^{\Phi_2(x)} \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz\,dy\,dx.$

This can also be written as $\int_a^b dx \int_{\Phi_1(x)}^{\Phi_2(x)} dy \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz.$

Similarly under suitable conditions a given triple integral can be expressed as an iterated integral in five other ways by permuting the variables.

**Example 9**

Evaluate $I = \int_0^a \int_0^x \int_0^y xyz\,dz\,dy\,dx$

**Solution**

$$I = \int_0^a \int_0^x \left[\frac{1}{2}xyz^2\right]_0^y dy\,dx$$

$$= \frac{1}{2}\int_0^a \int_0^x xy^3\,dy\,dx = \frac{1}{2}\int_0^a \left[\frac{1}{4}xy^4\right]_0^x dx$$

$$= \frac{1}{8}\int_0^a x^5\,dx = \frac{1}{8}\left[\frac{1}{6}x^6\right]_0^a$$

$$= \frac{a^6}{48}$$

**Example 10**

Evaluate $I = \int_0^{\log a} \int_0^x \int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

**Solution**

$$I = \int_0^{\log a} \int_0^x [e^{x+y+z}]_0^{x+y}\,dy\,dx$$

$$= \int_0^{\log a} \int_0^x \left[e^{2(x+y)} - e^{x+y}\right] dy\,dx$$

$$= \int_0^{\log a} \left[\frac{1}{2}e^{2(x+y)} - e^{x+y}\right]_0^x dx$$

$$= \int_0^{\log a} \left(\frac{1}{2}e^{4x} - \frac{3}{2}e^{2x} + e^x\right) dx$$

$$= \left[\frac{1}{8}e^{4x} - \frac{3}{4}e^{2x} + e^x\right]_0^{\log a}$$

$$= \frac{1}{8}a^4 - \frac{3}{2}a^2 + a - \frac{3}{8}$$

**Example 11**

Evaluate $I = \iiint_D \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ where $D$ is the region bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$.

**Solution**

The given region is a tetrahedron. The projection of the given region in the $x - y$ plane is the triangle bounded by the lines $x = 0, y = 0$ and $x + y = 1$ as the shown in the figure.

In the given region $x$ varies from 0 to 1. For each fixed $x, y$ varies from 0 to $1 - x$. For each fixed $(x, y), z$ varies from 0 to $1 - x - y$.

$$\therefore I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dx\,dy\,dz}{(x+y+z+1)^3}$$

$$= -\frac{1}{2}\int_0^1 \int_0^{1-x} [(x+y+z+1)^{-2}]_0^{1-x-y} dy\,dx$$

$$= -\frac{1}{2}\int_0^1 \int_0^{1-x} \left[\frac{1}{4} - (x+y+1)^{-2}\right] dy\,dx$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{1}{4}y + (x+y+1)^{-1}\right]_0^{1-x} dx$$

$$= -\frac{1}{2}\int_0^1 \left\{\frac{1}{4}(1-x) + \frac{1}{2} - (x+1)^{-1}\right\} dx$$

$$= -\frac{1}{2}\left[\frac{1}{4}x - \frac{1}{8}x^2 + \frac{1}{2}x - \log(x+1)\right]_0^1$$

$$= \frac{1}{2}\log 2 - \frac{5}{16}.$$

**Exercise 2**

1. Evaluate the following triple integrals.

(i) $\int_0^1 \int_0^1 \int_{\sqrt{x^2+y^2}}^2 xyz\,dz\;dy\,dx$

(ii) $\int_0^1 dx \int_0^2 dy \int_1^2 x^2yz\,dz$

(iii) $\int_0^{\pi} \int_0^{\pi/2} \int_0^k r^2 \sin\theta\;dr\,d\theta\,d\varphi$

2. Evaluate $\iiint_D (x^2 + y^2 + z^2)dx\,dy\,dz$ where $D$ is the region bounded by the planes $x + y + z = a; x = 0; y = 0$ and $z = 0$.

**Answers**

1. (i) $\frac{3}{8}$ (ii) 1 (iii) $\frac{1}{3}k^3\pi$ 2. $\frac{1}{20}a^5$

## 5.4 Change of Variables in double and triple integrals

The evaluation of a double or a triple integral sometimes becomes easier when we transform the given variables into new variables.

We state without proof the following theorem regarding change of variables in double and triple integrals.

**Theorem 1**

Consider a transformation given by the equation $x = x(u, v)$ and $y = y(u, v)$ where $x$ and $y$ have continuous first order partial derivatives. Let the region $D$ in the $x - y$ plane be mapped into the region $D^*$ in the $u - v$ plane. Further we assume that the Jacobian of the transformation $J \neq 0$ for all points in $D$. Then $\iint_D f(x, y)dx\, dy = \iint_{D^*} f[x(u, v), y(u, v)]|J|du\, dv$.

Similarly for triple integrals we have

$\iiint_D f(x, y)dx\, dy = \iiint_{D^*} f[x(u, v, w), y(u, v, w), z(u, v, w)]|J|du\, dv\, dw$

**Example 12**

Evaluate $I = \iint_D \frac{xydxdy}{\sqrt{x^2+y^2}}$ by transforming to polar coordinates where $D$ is the region enclosed by the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = 4a^2$ in the quadrant.

**Solution**

Put $x = r\cos\theta$ and $y = r\sin\theta$

We know that $J = r$.

Further in the given domain $D$, $0 \leq \theta \leq \pi/2$ and $a \leq r \leq 2a$.

$$\therefore I = \int_0^{\pi/2} \int_a^{2a} \left(\frac{r\cos\theta r\sin\theta}{r}\right) r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \cos\theta \sin\theta \left[\frac{1}{3}r^3\right]_a^{2a} d\theta$$

$$= \frac{7a^3}{3}\int_0^{\pi/2} \cos\theta \sin\theta \;\, d\theta$$

$$= \frac{7a^3}{3}\int_0^{\pi/2} \sin\theta \;\, d(\sin\theta)$$

$$= \frac{7a^3}{6}[\sin^2\theta]_0^{\pi/2}$$

$$= \frac{7a^3}{6}$$

**Example 13**

Evaluate the improper integral $I = \int_0^\infty e^{-x^2}dx$.

**Solution**

$$I^2 = II = \left(\int_0^\infty e^{-x^2}dx\right)\left(\int_0^\infty e^{-y^2}dy\right)$$

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dx\, dy$$

Put $x = r\cos\theta$ and $y = r\sin\theta$. Hence $J = r$.
The region of integration is the entire first quadrant.
Hence $r$ varies from $0\ to\ \infty$ and $\theta$ varies from $0\ to\ \pi/2$.

$$\therefore I^2 = \int_0^{\infty}\int_0^{\pi/2} e^{-r^2} r\, d\theta\, dr = \frac{\pi}{2}\int_0^{\infty} e^{-r^2}\, r\, dr.$$

$$= \frac{\pi}{2}\int_0^{\infty} -\frac{1}{2}e^{-r^2}\, d(-r^2) = \frac{\pi}{2}\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty}$$

$$= \frac{\pi}{2}\left(\frac{1}{2}\right) = \frac{\pi}{4}$$

$$\therefore I = \frac{\sqrt{\pi}}{2}$$

**Example 14**

Prove that $I = \iint_D \left(\frac{1-x^2-y^2}{1+x^2+y^2}\right)^{1/2} dx\, dy = \frac{\pi}{4}\left(\frac{\pi}{2} - 1\right)$ where D is the positive quadrant of the circle $x^2 + y^2 = 1$

**Solution**

Put $x = r\cos\theta$ and $y = r\sin\theta$.
$\therefore J = r$.
Further in D, $0 \le r \le 1$ and $0 \le \theta \le \pi/2$

$$\therefore I = \int_0^1\int_0^{\pi/2} \left(\frac{1-r^2}{1+r^2}\right)^{1/2} r\, d\theta\, dr$$

$$= \frac{\pi}{2}\int_0^1 \left(\frac{1-r^2}{1+r^2}\right)^{1/2} r\, dr$$

$$= \frac{\pi}{2}\int_0^1 \frac{1-r^2}{\sqrt{1-r^4}}\, r\, dr$$

$$= \frac{\pi}{4}\int_0^1 \frac{1-t}{\sqrt{1-t^2}}\, dt \qquad \text{(by putting } r^2 = t\text{)}$$

$$= \frac{\pi}{4}\left[\sin^{-1} t + (1-t^2)^{\frac{1}{2}}\right]_0^1$$

$$= \frac{\pi}{4}\left(\frac{\pi}{2} - 1\right)$$

**Example 15**

Evaluate $\iint_D \sqrt{x^2 + y^2}\, dx\, dy$ where $D$ is the parallelogram bounded by the lines $x + y = 0$; $x + y = 1$; $2x - 3y = 0$ and $2x - 3y = 4$.

**Solution**

Put $x + y = u$ and $2x - 3y = v$.
Then $J = -\frac{1}{5}$

Also $D$ is transformed into the rectangle bounded by the lines $u = 0; u = 1; v = 0$ and $v = 4$.

$\therefore I = \int_0^1 \int_0^4 \sqrt{u}\left(-\frac{1}{5}\right) dv\, du = -\frac{1}{5}\int_0^1 \sqrt{u}[v]_0^4\, du$

$= -\frac{4}{5}\left[\frac{2}{3}u^{3/2}\right]_0^1$

$= -\frac{8}{15}$

**Example 16**

Evaluate $I = \iiint_D xyz\, dx\, dy\, dz$ where D is the positive octant of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

**Solution**

Put $x = au, y = bv$ and $z = cw$

$\therefore J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$

Let $D^*$ be the image of D under the above transformation. Then $D^*$ is the positive octant of the sphere $u^2 + v^2 + w^2 = 1$.

$\therefore I = \iiint_{D^*} abc\, uvw\, abc\, du\, dv\, dw$

$= a^2b^2c^2 \iiint_{D^*} uvw\, du\, dv\, dw$

Now, put $u = r \sin\theta \cos\varphi$

$v = r \sin\theta \sin\varphi$

$w = r \cos\theta$

Then $J = r^2 \sin\theta$

$\therefore I = a^2b^2c^2 \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} r^5 \sin^3\theta \cos\theta \cos\varphi \sin\varphi\ d\varphi\, d\theta\, dr$

$= a^2b^2c^2 \int_0^1 r^5 dr \int_0^{\pi/2} \sin^3\theta \cos\theta\ d\theta \int_0^{\pi/2} \sin\varphi \cos\varphi\ d\varphi$

$= a^2b^2c^2 \left[\frac{1}{6}r^6\right]_0^1 \left[\frac{1}{4}\sin^4\theta\right]_0^{\pi/2} \left[\frac{1}{2}\sin^2\varphi\right]_0^{\pi/2}$

$= \frac{a^2b^2c^2}{48}$

**Exercise 3**

1. Evaluate the following double integrals using change of variables or otherwise over the region indicated.

(i) $\iint_D \sqrt{(x^2+y^2)}\, dx\, dy$ ; $D$ is the region bounded by the circle $x^2 + y^2 = a^2$.

(ii) $\iint_D \sqrt{(x^2+y^2)}\, dx\, dy$ ; $D$ is the region in the $x-y$ plane bounded by $x^2+y^2=4$ and $x^2+y^2=9$.

2. By transforming into polar coordinates evaluate.

(i) $\int_0^a \int_0^a \frac{x^2 dx\, dy}{(x^2+y^2)^{3/2}}$ (ii) $\int_0^1 \int_0^{\sqrt{x-x^2}} \frac{4xy}{x^2+y^2}\, e^{-(x^2+y^2)}\, dy\, dx$

3. Prove that $\iiint_D xyz(x^2+y^2+z^2)^{n/2}\, dx\, dy\, dz$ where $D$ is the positive octant of the sphere $x^2+y^2+z^2=a^2$ is $\frac{a^{n+6}}{8(n+6)}$ where $n+5>0$.

4. Evaluate $\iiint_D xyz\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) dx\, dy\, dz$ where $D$ is the positive octant of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

**Answers**

1. (i) $\frac{2}{3}\pi a^3$ (ii) $\frac{38}{3}\pi$

2. (i) $a/\sqrt{2}$ (ii) $1/e$

**BETA AND GAMMA FUNCTIONS**

**5.5 Beta and Gamma functions**

**Definition**

The Beta function is defined by

$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \quad (m,n>0)$.

The Gamma function is defined by

$\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx \quad (n>0)$.

**Theorem 2**

The Beta function $\beta(m,n)$ converges if $m,n>0$.

**Proof**

Let $I = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx = I_1 + I_2$ where $I_1 = \int_0^{1/2} x^{m-1}(1-x)^{n-1}\, dx$

We first consider $I_1$.

When $m \geq 1$ it is a proper integral and hence $I$ converges.

When $m<1$ the function $f(x) = x^{m-1}(1-x)^{n-1}$ has an infinite discontinuity at $x=0$.

Now $\lim_{x\to 0} x^{1-m} f(x) = \lim_{x\to 0}(1-x)^{n-1} = 1$.
Hence by $\mu$-test, $I_1$ is convergent if $1 - m < 1$.
(i.e.) $I_1$ is convergent if $m > 0$.
Similarly $I_2$ is convergent if $n > 0$.
Hence $I = I_1 + I_2$ converges when $m > 0$ and $n > 0$.

**Theorem 3**

The Gamma function $\Gamma(n)$ converges if $n > 0$.

**Proof**

Let $I = \int_0^\infty e^{-x} x^{n-1} dx \quad (n > 0)$

$= I_1 + I_2$ where $I_1 = \int_0^a e^{-x} x^{n-1}\, dx$ and $I_2 = \int_a^\infty e^{-x} x^{n-1}\, dx$ and $a > 0$.

First we consider $I_1$.

When $n \geq 1$ it is a proper integral and hence $I_1$ converges. When $n < 1, f(x) = e^{-x} x^{n-1}$ has an infinite discontinuity at $x = 0$.

In this case $\lim_{x\to 0} x^{1-n} f(x) = \lim_{x\to 0} e^{-x} = 1$.
Hence by $\mu$-test $I_1$ is convergent if $1 - n < 1$.
(i.e.) $I_1$ is convergent if $n > 0$.
Now we consider $I_2$.
When $x > 0, e^x > \frac{x^r}{r!}$ for any positive integer $r$.

$\therefore e^{-x} < \frac{r!}{x^r}$

$\therefore e^{-x} x^{n-1} < \frac{r!}{x^{r-n+1}}$

Whenever $n$ may be, we can choose $r$ such that $r - n + 1 > 1$.
With this choice of $r$, $\int_a^\infty \frac{dx}{x^{r-n+1}}$ is convergent.
Hence by comparison test $I_2$ is convergent.
Hence $I = I_1 + I_2$ converges when $n > 0$.

**Properties and results involving Beta and Gamma Functions**

**1.** $\boldsymbol{\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx}$

**Proof**

Put $x = \frac{y}{1+y}$. Hence $y = \frac{x}{1-x}$.

When $x = 0, y = 0$. When $x \to 1, y \to \infty$.

Also $dx = \frac{dy}{(1+y)^2}$.

$\therefore \beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}(1+y)^{n-1}(1+y)^2} dy$

$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$

$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx.$

**2.** $\boldsymbol{\beta(m,n) = 2\int_0^{\pi/2} (\sin x)^{2m-1} (\cos x)^{2n-1}\, dx}$

**Proof**

Put $x = \sin^2 t$.

When $x = 0; t = 0$ and when $x = 1; t = \pi/2$.

$\therefore \beta(m,n) = \int_0^{\pi/2} (\sin^2 t)^{m-1} (\cos^2 t)^{n-1} 2 \sin t \cos t\, dt$

$= 2\int_0^{\pi/2} (\sin x)^{2m-1} (\cos x)^{2n-1}\, dx.$

**3.** $\boldsymbol{\beta(m,n) = \beta(n,m)}$

**Proof**

$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx.$

Put $x = 1 - y$.

$\therefore$ When $x = 0, y = 1$ and when $x = 1, y = 0$.

$\therefore \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

$= \int_0^1 (1-y)^{m-1} y^{n-1}\, (-dy)$

$= \int_0^1 y^{n-1}(1-y)^{m-1}\, dy$

$= \beta(n,m).$

**4.** $\boldsymbol{\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)}$

**Proof**

$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

$= \int_0^1 x^{m-1}(1-x)^{n-1}(x + 1 - x)\, dx$

$= \int_0^1 x^m (1-x)^{n-1}\, dx + \int_0^1 x^{m-1}(1-x)^n\, dx$

$= \beta(m+1,n) + \beta(m,n+1).$

**5.** $\boldsymbol{\Gamma(n+1) = n\Gamma(n)}$

**Proof**

$$\Gamma(n+1) = \int_0^{\infty} x^n e^{-x} dx$$
$$= \lim_{a\to\infty}\left[\int_0^a x^n e^{-x} dx\right] = \lim_{a\to\infty}\left[-\int_0^a x^n d(e^{-x})\right]$$
$$= \lim_{a\to\infty}\left[[-x^n e^{-x}]_0^a + \int_0^a n e^{-x} x^{n-1}\, dx\right]$$
$$= \lim_{a\to\infty}[-x^n e^{-a}]_0^a + n\Gamma(n)$$
$$= n\Gamma(n) \qquad \left[since\ \lim_{a\to\infty}(-x^n e^{-a}) = 0\right]$$

**6.** $\boldsymbol{\Gamma(1) = 1}$

**Proof**

$$\Gamma(1) = \int_0^{\infty} x e^{-x} dx$$
$$= \lim_{a\to\infty}\left[[-xe^{-x}]_0^a + \int_0^a e^{-x}\, dx\right]$$
$$= \lim_{a\to\infty}[-e^{-x}]_0^a$$
$$= \lim_{a\to\infty}[-e^{-a} + 1]$$
$$= 1.$$

**7.** $\boldsymbol{\Gamma(n+1) = n!}$ **where** $\boldsymbol{n}$ **is a positive integer.**

**Proof**

We have $\Gamma(n+1) = n\Gamma(n)$ (by 5)

$$= n(n-1)\Gamma(n-1)$$
$$= n(n-1)\ldots 2.1.\Gamma(1)$$
$$= n! \qquad \text{(using 6)}$$

**8.** $\boldsymbol{\Gamma(n) = 2\int_0^{\infty} e^{-y^2} y^{2n-1}\, dy}$

**Proof**

We have $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$

Put $x = y^2$. Hence $dx = 2y\, dy$.

$$\therefore\ \Gamma(n) = \int_0^{\infty} e^{-y^2} (y^2)^{n-1} 2y\, dy$$
$$= 2\int_0^{\infty} e^{-y^2} y^{2n-1}\, dy$$

**9.** $\boldsymbol{\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$

**Proof**

We have (by 8) $\Gamma(m) = 2\int_0^\infty e^{-y^2} y^{2m-1}\,dy$ and

$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\,dx$

$\therefore\ \Gamma(m)\Gamma(n) = 4\int_0^\infty e^{-y^2} y^{2m-1}\,dy \int_0^\infty e^{-x^2} x^{2n-1}\,dx$

$\qquad = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, x^{2n-1}\,dx\,dy.$

Put $x = r\cos\theta$ and $y = r\sin\theta$. Hence $|J| = r$.

Further the region of integration is the entire first quadrant and hence $r$ varies from 0 to ∞ and $\theta$ varies from 0 to $\pi/2$.

$\therefore\ \Gamma(m)\Gamma(n) = 4\int_0^\infty \int_0^{\pi/2} e^{-r^2}\, r^{2m+2n-1}(\cos\theta)^{2n-1}(\sin\theta)^{2m-1}\,d\theta\,dr$

$= 4\int_0^\infty e^{-r^2} r^{2m+2n-1}\,dr \int_0^{\pi/2}(\cos\theta)^{2n-1}(\sin\theta)^{2m-1}\,d\theta$

$= 4\int_0^\infty e^{-r^2}(r^2)^{m+n-1}\frac{1}{2}d(r^2)\int_0^{\pi/2}(\cos\theta)^{2n-1}(\sin\theta)^{2m-1}\,d\theta$

$= 4\left[\frac{1}{2}\Gamma(m+n)\right]\left[\frac{1}{2}\beta(m,n)\right]$ (using (2))

$= \Gamma(m+n)\beta(m,n)$ .

$\therefore\ \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

**10.** $\boldsymbol{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$

**Proof**

We have $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\,dx$ (using 8)

$\qquad = 2\int_0^\infty e^{-x^2} x^{2(1/2)-1}\,dx$

$\therefore\ \Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}\,dx = 2\left(\frac{\sqrt{\pi}}{2}\right)$

$\qquad = \sqrt{\pi}$ .

**Alter** We know that $\beta(m,n) = \int_0^{\pi/2}(\sin x)^{2m-1}(\cos x)^{2n-1}\,dx$

$\therefore\ \beta\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\pi/2} dx = 2[x]_0^{\pi/2} = \pi$

$\therefore\ \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \pi$ (using (9))

$\therefore\ [\Gamma(1/2)]^2 = \pi$

$\therefore\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

**11.** $\boldsymbol{\Gamma(n) = \int_0^1 [\log(1/x)]^{n-1}\,dx}$

**Proof**

We have that $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx$.

Put $x = \log(1/y)$. Hence $dx = -(1/y)dy$

When $x = 0, y = 1$ and when $x = \infty, y = 0$.

$$\therefore \Gamma(n) = \int_1^0 y\left[\log\left(\frac{1}{y}\right)\right]^{n-1}\left(-\frac{1}{y}\right)dy$$

$$= \int_0^1 \left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx\,.$$

**12. $2^{2n-1}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) = \Gamma(2n)\sqrt{\pi}$**

**(This is known as the duplication formula)**

**Proof**

Let $I = \int_0^{\pi/2} \sin^{2n} x\ dx$

We notice that $\int_0^{\pi/2} \sin^{2n} 2x\ dx = I$

For, $\int_0^{\pi/2} \sin^{2n} 2x\ dx = \frac{1}{2}\int_0^{\pi} \sin^{2n} y\ dy$ (putting $2x = y$)

$$= \int_0^{\pi/2} \sin^{2n} y\ dy$$

(since $\sin^{2n}(\pi - y) = \sin^{2n} y$)

$$= I$$

Taking $I = \int_0^{\pi/2} \sin^{2n} x\ dx = \int_0^{\pi/2} (\sin x)^{2\left(n+\frac{1}{2}\right)-1}(\cos x)^{2\left(\frac{1}{2}\right)-1}\,dx$

$$= \frac{1}{2}\beta\left(n+\frac{1}{2},\frac{1}{2}\right) \quad \text{(by 2)}$$

$$= \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(n+1)} = \frac{\Gamma\left(n+\frac{1}{2}\right)\sqrt{\pi}}{2\Gamma(n+1)} \quad (1)$$

Now taking $I = \int_0^{\pi/2} \sin^{2n} 2x\ dx$

$$= \int_0^{\pi/2} 2^{2n}(\sin x)^{2n}(\cos x)^{2n}\,dx$$

$$= 2^{2n}\left[\frac{1}{2}\beta\left(n+\frac{1}{2}, n+\frac{1}{2}\right)\right] \quad \text{(by (2))}$$

$$= 2^{2n-1}\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n+1)} \quad (2)$$

From (1) and (2) we get

$$\frac{\Gamma\left(n+\frac{1}{2}\right)\sqrt{\pi}}{2\Gamma(n+1)} = 2^{2n-1}\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n+1)}$$

$$\therefore \Gamma(2n+1)\sqrt{\pi} = 2^{2n}\Gamma(n+1)\Gamma\left(n+\frac{1}{2}\right).$$

$$\therefore 2n\,\Gamma(2n)\sqrt{\pi} = 2^{2n}n\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) \quad \text{(using (5))}$$

$\therefore\ 2^{2n-1}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) = \Gamma(2n)\sqrt{\pi}.$

**13. $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\pi$**

**Proof**

Put $n = \frac{1}{4}$ in the duplication formula.

$$\therefore\ \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\sqrt{\pi}}{2^{-\frac{1}{2}}}$$
$$= \sqrt{2}\pi$$

**Example 17**

Evaluate $\int_0^\infty x^6 e^{-3x} dx$

**Solution**

Put $y = 3x$. Hence $dy = 3\ dx$.

$$\text{Now, } \int_0^\infty x^6 e^{-3x} dx = \int_0^\infty \left(\frac{y}{3}\right)^6 e^{-y}\left(\frac{dy}{3}\right)$$
$$= \left(\frac{1}{3}\right)^7 \int_0^\infty y^6 e^{-y} dy = \left(\frac{1}{3}\right)^7 \Gamma(7)$$
$$= \left(\frac{1}{3}\right)^7 6! = \frac{80}{243}.$$

**Example 18**

Prove that $\int_0^\infty \frac{e^{-st}}{\sqrt{t}}\ dt = \sqrt{\pi/s}$ where $s > 0$

**Solution**

Put $st = u$. Hence $s\ dt = du$

$$\therefore\ \int_0^\infty \frac{e^{-st}}{\sqrt{t}}\ dt = \frac{1}{\sqrt{s}}\int_0^\infty e^{-u}u^{\left(-\frac{1}{2}\right)} du$$
$$= \frac{1}{\sqrt{s}}\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi/s}$$

**Example 19**

Evaluate $I = \int_0^1 x^4(1-x)^3 dx$

**Solution**

$$I = \int_0^1 x^{5-1}(1-x)^{4-1} dx$$
$$= \beta(5,4) = \frac{\Gamma(5)\Gamma(4)}{\Gamma(9)} = \frac{4!3!}{8!} = \frac{1}{280}$$

**Example 20**

Prove that $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$

**Solution**

$$\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \int_0^{\pi/2} (\sin\theta)^{\frac{1}{2}} (\cos\theta)^0 d\theta$$

$$= \int_0^{\pi/2} (\sin\theta)^{2\left(\frac{3}{4}\right)-1} (\cos\theta)^{2\left(\frac{1}{2}\right)-1} d\theta$$

$$= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{2}\right) \qquad \text{(by 2)}$$

$$= \frac{1}{2}\left[\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}\right] = \frac{1}{2}\left[\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}\right]$$

$$= 2\left[\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)}\right]$$

Now, $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \int_0^{\pi/2} (\sin\theta)^{\left(-\frac{1}{2}\right)} d\theta = \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$

$$= \frac{1}{2}\left[\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}\right]$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}} \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta = \left[\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{3}{4}\right)}\right] \times \left[\frac{2\,\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)}\right]$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi \qquad \text{(by 10)}$$

**Example 21**

Prove that $\int_0^{\pi/2} \sin^m x \cos^n x \; dx = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$. Hence find (i) $\int_0^{\pi/2} \sin^5 x \cos^6 x \; dx$ (ii) $\int_0^{\pi/2} \sin^6 x \cos^8 x \; dx$

**Solution**

We know $\beta(p, q) = 2\int_0^{\pi/2} (\sin x)^{2p-1} (\cos x)^{2q-1}\, dx$

Put $p = \frac{m+1}{2}$ and $q = \frac{n+1}{2}$

$\therefore 2p - 1 = m$ and $2q - 1 = n$

$\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right) = 2\int_0^{\pi/2} \sin^m x \cos^n x \; dx$

Hence $\int_0^{\pi/2} \sin^m x \cos^n x \; dx = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$

(i) $\int_0^{\pi/2} \sin^5 x \cos^6 x \; dx = \frac{1}{2}\beta\left(\frac{5+1}{2}, \frac{6+1}{2}\right) = \frac{1}{2}\beta\left(3, \frac{7}{2}\right)$

$$= \frac{1}{2}\left[\frac{\Gamma(3)\Gamma(7/2)}{\Gamma(13/2)}\right] = \frac{1}{2}\left[\frac{\Gamma(3)\Gamma(7/2)}{\frac{11}{2}.\frac{9}{2}.\frac{7}{2}.\Gamma(7/2)}\right] = \frac{1}{2}\left(\frac{2!2^3}{11.9.7}\right) = \frac{8}{693}$$

(ii) $\int_0^{\pi/2} \sin^6 x \cos^8 x \; dx = \frac{1}{2}\beta\left(\frac{6+1}{2}, \frac{8+1}{2}\right) = \frac{1}{2}\beta\left(\frac{7}{2}, \frac{9}{2}\right)$

$$= \frac{1}{2}\left[\frac{\Gamma(7/2)\Gamma(9/2)}{\Gamma(8)}\right] = \frac{1}{2}\left[\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)}{7!}\right]$$
$$= \frac{\pi}{2}\left[\frac{1.3.5.1.3.5.7}{2^7.7!}\right] = \frac{\pi}{2}\left[\frac{7!5!}{3.2^{14}.7!}\right]$$
$$= \frac{5\pi}{2^{12}}$$

**Example 22**

Prove that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$ using Gamma function.

**Solution**

Let $I = \int_0^\infty e^{-x^2}\,dx$

Put $x^2 = y$ so that $dx = \frac{dy}{2x} = \frac{dy}{2\sqrt{y}}$

$\therefore I = \int_0^\infty e^{-y}\frac{dy}{2\sqrt{y}} = \frac{1}{2}\int_0^\infty e^{-y}\,y^{\frac{1}{2}-1}dy = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$

**Exercise 4**

1. Evaluate (i) $\int_0^\infty x^6 e^{-3x}dx$ (ii) $\int_0^\infty x^2 e^{-x^2}dx$
2. Evaluate $\int_0^1 x^7(1-x)^8 dx$
3. Evaluate (i) $\int_0^1 x^2(1-x)^3 dx$ (ii) $\int_0^1 x^3(1-x^2)^{5/2}dx$
(iii) $\int_0^1 x^3\sqrt{1-x}\,dx$
4. Find the value of $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}dx$
5. Evaluate (i) $\int_0^{\pi/2}\cos^3 x \sin^6 x\ dx$ (ii) $\int_0^{\pi/2}\sin^3 x\cos^{10} x\ dx$

**Answers**

1. (i) $\frac{80}{243}$ (ii) $\frac{\sqrt{2}\pi}{16}$ 2. $\frac{7!8!}{16!}$ 3. (i) $\frac{1}{60}$ (ii) $\frac{1}{9}$ (iii) $\frac{2}{3}$
4. $\frac{1}{5005}$ 5. (i) $\frac{2}{63}$ (ii) $\frac{2}{143}$

## FOURIER SERIES

### 5.6 Fourier series

**Definition**

Let $f(x)$ be a bounded integrable function defined on $[-\pi,\pi]$. The trigonometric series $\frac{a_n}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$ where

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \ dx$

is called Fourier series of $f(x)$ and $a_n, b_n$ are called Fourier coefficients of $f(x)$

Fourier proved that for several functions $f(x)$, its Fourier series actually converges to $f(x)$.

**Note** If $f(x)$ is defined in an arbitrary interval $[\lambda, \lambda + 2\mu]$ of length $2\mu$,

### 5.6.1 The Cosine and Sine series

Let $f(x)$ be defined in the interval $[0, \pi]$. Define $f(x) = f(-x)$ if $-\pi \le x \le 0$. Then $f(x)$ is an even function in $[-\pi, \pi]$.

We know that $\sin nx$ is an odd function and $\cos nx$ is an even function. Hence $f(x)\sin nx$ is an odd function and $f(x)\cos nx$ is an even function.

$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \ dx = 0$ and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \ dx$

(i.e.) $a_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \ dx$

Hence the corresponding Fourier series of $f(x)$ is the cosine series given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \ dx$ where $a_n$ is given above.

Similarly if we define $f(x) = -f(x)$ if $-\pi \le x \le 0$ then $f(x)$ in an odd function in $[-\pi, \pi]$ and its Fourier series becomes the sine series $\sum_{n=1}^{\infty} b_n \sin nx$ where $b_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \ dx$.

**Example 23**

Determine the Fourier expansion of $f(x) = x$ where $-\pi < x < \pi$

**Solution**

Let $f(x) = x$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\, dx$

$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = 0$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \ dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx \ dx$

$= \frac{1}{\pi}\left[\left[\frac{x\sin nx}{n}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\sin nx \ dx\right]$

$$= \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{\pi}$$
$$= \frac{1}{n^2\pi}\left[\frac{\cos nx}{n}\right]_{-\pi}^{\pi}$$
$$= \frac{1}{n^2\pi}[\cos n\pi - \cos(-n\pi)]$$
$$= \frac{1}{n^2\pi}(\cos n\pi - \cos n\pi) = 0$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx \ dx$$
$$= \frac{1}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{\pi}$$
$$= -\frac{1}{n\pi}[\pi\cos n\pi + \pi\cos n\pi]$$
$$= -\frac{2\cos n\pi}{n}$$
$$= -\frac{2(-1)^n}{n}$$
$$= \frac{2(-1)^{n+1}}{n}$$
$$\therefore x = \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{2}{n}\right)\sin nx$$
$$\therefore x = 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right]$$

**Example 24**

If $f(x) = \begin{cases} -x \ if \ -\pi < x < 0 \\ \ \ x \ if \ 0 \le x < \pi \end{cases}$ expand $f(x)$ as a Fourier series the interval $(-\pi, \pi)$

**Solution**

Clearly $f(-x) = f(x)$ for all $x \in (-\pi, \pi)$.

$\therefore f(x)$ is an even function in $(-\pi, \pi)$.

$\therefore f(x)$ can be expanded as a Fourier series of the form $\frac{a_n}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx$$
$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\ dx \ \text{(since } f(x) \text{ is an even function)}$$
$$= \frac{2}{\pi}\int_{0}^{\pi} x\ dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\frac{\pi^2}{2}\right] = \pi.$$
$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx \ dx$$
$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi}\sin nx \ dx$$

$= \frac{2}{\pi n^2}[\cos n\pi]_0^\pi$

$= \frac{2}{\pi n^2}[(-1)^n - 1]$

$= \begin{cases} -\frac{4}{\pi n^2} & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{cases}$

$\therefore f(x) = \frac{\pi}{2} - \frac{\pi}{4}\sum\left(\frac{\cos nx}{n^2}\right)$ where $n$ is odd.

$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$

**Example 25**

Find the Fourier (i) cosine series (ii) sine series for the function $f(x) = \pi - x$ in $(0, \pi)$.

**Solution**

(i) Let $f(x) = \pi - x$.

The Fourier cosine series of $f(x)$ is given by

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

$a_0 = \frac{2}{\pi}\int_{-\pi}^{\pi}(\pi - x)\,dx$

$= \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^\pi$

$= \frac{2}{\pi}\left(\frac{x^2}{2}\right) = \pi.$

$a_n = \frac{2}{\pi}\int_0^\pi (\pi - x)\cos nx\ dx$

$= \frac{2}{\pi}\left[\left\{(\pi - x)\frac{\sin nx}{n}\right\}_0^\pi + \frac{1}{n}\int_0^n \sin nx\ dx\right]$

$= \frac{2}{\pi}\left[-\frac{\cos nx}{n^2}\right]_0^\pi = \frac{2}{\pi n^2}[(-1)^{n+1} + 1]$

$= \begin{cases} \frac{4}{\pi n^2} & when\ n\ is\ odd \\ 0 & when\ n\ is\ even \end{cases}$

$\therefore \pi - x = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$

(ii) Let $f(x) = \pi - x$. The Fourier sine series of $f(x)$ is given by $\sum_{n=1}^{\infty} b_n \sin nx$.

$b_n = \frac{2}{\pi}\int_0^\pi (\pi - x)\sin nx\ dx$

$= \frac{2}{\pi}[\left\{\frac{(\pi-x)\cos nx}{n}\right\}_0^\pi - \frac{1}{n}\int_0^\pi \cos nx\ dx$

$= \frac{2}{\pi}\left(\frac{\pi}{n}\right) - \frac{2}{\pi n^2}[\sin nx]_0^\pi = \frac{2}{n}$

$\therefore \pi - x = 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$

**Example 26**

If $f(x) = x$ is defined in $-l < x < l$ with period $2l$ find the Fourier expansion of $f(x)$

**Solution**

Since $f(x)$ is an odd function $a_n = 0$ for all $n \geq 0$.

Now, $b_n = \frac{2}{l}\int_0^l x \sin\left(\frac{n\pi x}{l}\right) dx$

$$= \frac{2}{l}\left[-\frac{lx}{n\pi}\cos\left(\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left(-\frac{l^2\cos n\pi}{n\pi}\right) = -\frac{2l(-1)^n}{n\pi}$$

$$= \frac{2(-1)^{n+1}l}{n\pi}.$$

$\therefore$ The Fourier series is $x = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}l}{n}\sin\left(\frac{n\pi x}{l}\right)\right]$

**Example 27**

Find the half range Fourier sine series of $f(x) = x$ in $0 < x < 2$.

**Solution**

The Fourier sine series for $f(x)$ in $(0, 2)$ is given by

$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$ where $b_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right) dx$

$$= \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \left[-\frac{x2\cos\left(\frac{n\pi x}{2}\right)}{n\pi} + \frac{4\sin\left(\frac{n\pi x}{2}\right)}{n^2\pi^2}\right]_0^2$$

$$= \left(-\frac{4\cos n\pi}{n\pi}\right) = -\frac{4}{\pi}\left[\frac{(-1)^n}{n}\right]$$

$\therefore$ The Fourier sine series for $f(x) = x$ is given by

$x = -\frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{2}\right)\right]$.

**Exercise 5**

1. Find the Fourier series to represent $f(x)$ in $(-\pi, \pi)$

(i) $f(x) = \begin{cases} -1 & in -\pi < x \leq 0 \\ 1 & in\ \ 0 \leq x \leq \pi \end{cases}$

(ii) $f(x) = \begin{cases} 1 & in -\pi < x \leq 0 \\ -2 & in\ \ 0 < x \leq \pi \end{cases}$

2. If $f(x) = \begin{cases} 0 & in -\pi < x \leq 0 \\ x & in \ 0 < x < \pi \end{cases}$ prove that its Fourier series is $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots\right] + \left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \cdots\right]$. Hence prove that $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

3. Expand the function $y = \cos 2x$ in a series of sines in the interval $(0, \pi)$.

**Answers**

1. (i) $f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$

(ii) $-\frac{1}{2} - \frac{6}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}$

3. $-\frac{4}{\pi}\left[\frac{\sin x}{2^2-1} + \frac{3\sin 3x}{2^2-3^2} + \frac{5\sin 5x}{2^2-5^2} + \cdots\right]$